A LEMMA ON FLIPS (PRIVATE NOTE)

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The following lemma is missing in the literature.

Lemma. *When we consider* 4*-fold flipping contraction* $f : (X, D) \to Z$*, we can assume that there exists a closed point* $P \in Z$ *such that a flip of* f *exists outside* P *after shrinking* Z *suitably.*

Proof. Let $f : (X, D) \to Z$ be a flipping contraction with $\dim X = 4$. This means that f is small, $-(K_X + D)$ is f-ample, Z is normal, and (X, D) is dlt. We assume that D is a $\mathbb{Q}$-divisor. For our purpose, we can assume that Z is affine without loss of generality. Let r be a positive integer such that $r(K_X + D)$ is Cartier. Let H be a *sufficiently* general hypersurface on Z such that H does not contain any associated primes of $R^1 f_*\mathcal{O}_X(mr(K_X + D))$ for all $m > 0$. We put $S = f^*H = f_*^{-1}H$. Then $(X, S + D)$ is dlt and $K_S + B = (K_X + S + D)|_S$ is also dlt. Note that $f : (S, B) \to H$ is a flipping contraction. By the choice of H,

$$f_*\mathcal{O}_X(mr(K_X + S + D)) \to f_*\mathcal{O}_S(mr(K_S + B)) \to 0$$

for all $m \geq 0$ since $R^1 f_*\mathcal{O}_X = 0$. Note that $\bigoplus_{m\geq 0} f_*\mathcal{O}_S(mr(K_S+B))$ is finitely generated since $\dim S = 3$. By taking truncation and assuming that r is sufficiently large, we can assume that $\bigoplus_{m\geq 0} f_*\mathcal{O}_S(mr(K_S + B))$ is generated by $f_*\mathcal{O}_S(r(K_S + B))$. We consider the $\mathcal{O}_Z$-subalgebra $\mathcal{R}$ of $\bigoplus_{m\geq 0} f_*\mathcal{O}_X(mr(K_X+S+D))$ generated by $f_*\mathcal{O}_X(r(K_X+S+D))$. We define $g : \mathrm{Proj}_Z\mathcal{R} \to Z$. If we restrict g to H, then we obtain the flip of $f : (S, B) \to H$. Therefore, g is small in a neighborhood of H. Thus, we can assume that g is small by shrinking Z around H. Let X^+ be the normalization of $\mathrm{Proj}_Z\mathcal{R}$. It is not difficult to see that $X^+ \to Z$ is a flip of $f : (X, D) \to Z$. It immediately implies the lemma. □

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