AN EXAMPLE OF TORIC FLOPS (PRIVATE NOTE)

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Abstract. We construct an example of global toric 3-dimensional terminal flops that has interesting properties.

Contents

1. Introduction

We explain examples of toric contraction morphisms. There are no theorems in this paper. The main purpose is to construct an example of 3-dimensional (global) toric terminal flops that has interesting properties. We describe it in details. We treat non-$\mathbb{Q}$-factorial toric varieties. So, various new phenomena happen even in the toric category. For the toric Mori theory for non-$\mathbb{Q}$-factorial varieties, see [Fj]. We use the same notation as in [Fj] and [FS].

2. An example of toric flops

Example 2.1 (Global toric 3-dimensional terminal flop)**.** We have the following toric flopping diagram;

$$\begin{array}{ccc} X & \dashrightarrow & X^+ \\ & \searrow \quad \swarrow & \\ & W & \end{array}$$

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This paper grew out of the second author's handwritten pictures.

such that

(1) X, X^+ and W are all projective toric 3-folds,
(2) $\rho(X/W) = \rho(X^+/W) = 1$, $\rho(X) = 4$, and $\rho(W) = 3$,
(3) K_X (resp. K_{X^+}) is Cartier and φ-numerically trivial (resp. φ^+-numerically trivial), where $\varphi : X \longrightarrow W$ (resp. $\varphi^+ : X^+ \longrightarrow W$) is a small toric morphism,
(4) X, X^+ and W have only terminal singularities, and
(5) $\mathrm{Exc}(\varphi) = \mathbb{P}^1 \amalg \mathbb{P}^1$ and $\mathrm{Exc}(\varphi^+) = \mathbb{P}^1 \amalg \mathbb{P}^1$.

More precisely,

(6) both $\mathrm{Sing}X$ and $\mathrm{Sing}X^+$ are only one ordinary double point, where $\mathrm{Sing}X$ (resp. $\mathrm{Sing}X^+$) is the singular locus of X (resp. X^+). In particular, X and X^+ are not $\mathbb{Q}$-factorial,
(7) the flop $X \dashrightarrow X^+$ is the union of two *simplest flops*, where the simplest flop means the flop described in [Fl, p.49–p.50]. So, W has three ordinary double points,
(8) let P be the ordinary double point on X. Then $P \cap \mathrm{Exc}(\varphi) = \emptyset$. Thus φ is an isomorphism around P. We put $X^0 := X \setminus P$ and $W^0 := W \setminus \varphi(P)$. Then X^0 is non-singular and $\rho(X^0/W^0) = 2$.
(9) the flop $X \dashrightarrow X^+$ factors as follows:

$$\begin{array}{ccccccc} X & & \dashrightarrow & Z & \dashrightarrow & & X^+ \\ & \searrow & & \swarrow \quad \searrow & & \swarrow & \\ & & V_1 & & V_2 & & \end{array}$$

Each step is the simplest flop. Every morphism is over W. We note that V_1, V_2 and Z are not projective over W. However, every variety is projective over W^0.

Note that the flopping locus is irreducible by Reid's description when X is $\mathbb{Q}$-factorial (see [R, (2.5) Corollary]). This example shows that it is difficult to study the behaviors of the toric contraction morphisms without $\mathbb{Q}$-factoriality. In this example, the flopping locus is contained in a non-singular open subset.

3. Construction

3.1. We fix $N \simeq \mathbb{Z}^3$. Let e_1, e_2 and e_3 be the standard basis of $\mathbb{Z}^3$. We put

$$e_4 = e_1 + e_2 + e_3 = (1, 1, 1),$$
$$e_5 = e_3 + e_4 = (1, 1, 2),$$
$$e_6 = e_1 + e_4 = (2, 1, 1),$$
$$e_7 = e_2 + e_4 = (1, 2, 1).$$

3.2. We consider the fan Δ_1:

$\Delta_1 = \{\langle e_1, e_2, e_6\rangle, \langle e_2, e_3, e_5\rangle, \langle e_2, e_5, e_6\rangle, \langle e_1, e_3, e_5, e_6\rangle, \text{and their faces}\}$.

The picture is as follows (see Figure 1):

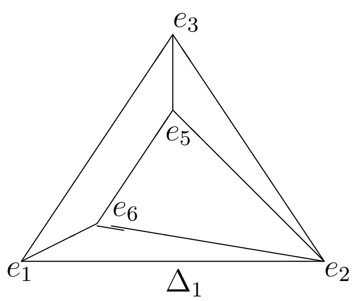


FIGURE 1

We put $\Delta_Y = \{\langle e_1, e_2, e_3\rangle, \text{and its faces}\}$ and $Y := X(\Delta_Y)$ (see Figure 2).

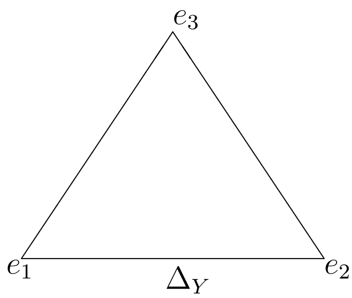


FIGURE 2

Then $f_1 : X_1 := X(\Delta_1) \longrightarrow Y$ has the following properties:

(1-i) X_1 is projective over Y,
(1-ii) X_1 has only canonical (not terminal) singularities,
(1-iii) $-K_{X_1}$ is ample over Y,
(1-iv) $\rho(X_1/Y) = 1$, and
(1-v) f_1 contracts a reducible divisor to a point.

The ampleness of $-K_{X_1}$ follows from the convexity of the roof of the shed of Δ_1 (see [R, (4.5) Proposition]). So, this is also an example of non-$\mathbb{Q}$-factorial divisorial contraction (see [Fj, Example 4.1]).

3.3. We consider

$$\Delta_2 = \left\{ \begin{array}{cccc} \langle e_1, e_3, e_5, e_6\rangle, & \langle e_2, e_3, e_5\rangle, & \langle e_1, e_2, e_6\rangle, & \langle e_2, e_6, e_7\rangle, \\ \langle e_2, e_5, e_7\rangle, & \langle e_5, e_6, e_7\rangle, & \text{and their faces} & \end{array} \right\},$$

and

$$\Delta_3 = \left\{ \begin{array}{ccc} \langle e_1, e_3, e_5, e_6 \rangle, & \langle e_2, e_3, e_5 \rangle, & \langle e_1, e_2, e_6 \rangle, \\ \langle e_2, e_6, e_7 \rangle, & \langle e_2, e_5, e_7 \rangle, & \langle e_4, e_5, e_6 \rangle, \\ \langle e_4, e_6, e_7 \rangle, & \langle e_4, e_5, e_7 \rangle, & \text{and their faces} \end{array} \right\}.$$

Then $f_2 : X_2 := X(\Delta_2) \longrightarrow X_1$ is a divisorial contraction such that

(2-i) $-K_{X_2}$ is f_2-ample,

(2-ii) $\rho(X_2/X_1) = 1$,

(2-iii) X_2 has log-terminal (not canonical) singularities.

See Figure 3 below.

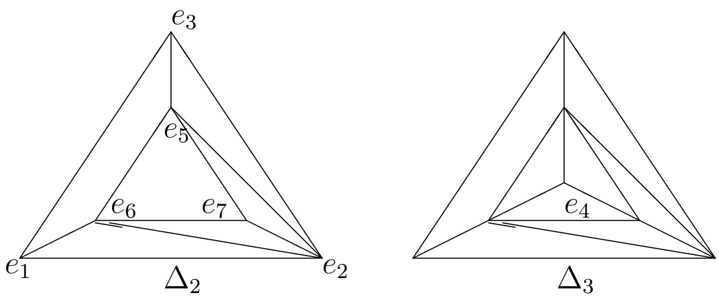


FIGURE 3

The morphism $f_3 : X_3 := X(\Delta_3) \longrightarrow X_2$ is also a divisorial contraction. It has the following properties:

(3-i) SingX_3 is only one ordinary double point. In particular, X_3 has non-$\mathbb{Q}$-factorial terminal singularities,

(3-ii) K_{X_3} is f_3-ample, and

(3-iii) $\rho(X_3/X_2) = 1$.

We note that $\rho(X_3/Y) = 3$.

3.4. We consider the following fans:

$$\Delta_4 = \left\{ \begin{array}{ccc} \langle e_1, e_2, e_6, e_7 \rangle, & \langle e_1, e_3, e_5, e_6 \rangle, & \langle e_2, e_3, e_5, e_7 \rangle, \\ \langle e_5, e_6, e_7 \rangle, & \text{and their faces} & \end{array} \right\},$$

and

$$\Delta_5 = \left\{ \begin{array}{cccc} \langle e_1, e_2, e_6, e_7 \rangle, & \langle e_1, e_3, e_5, e_6 \rangle, & \langle e_2, e_3, e_5, e_7 \rangle, & \langle e_4, e_5, e_6 \rangle, \\ \langle e_4, e_6, e_7 \rangle, & \langle e_4, e_5, e_7 \rangle, & \text{and their faces} & \end{array} \right\}.$$

We put $f_5 : X_5 := X(\Delta_5) \longrightarrow X_4 := X(\Delta_4)$ and $\varphi : X_3 \longrightarrow X_5$. The pictures are as follows (see Figure 4):

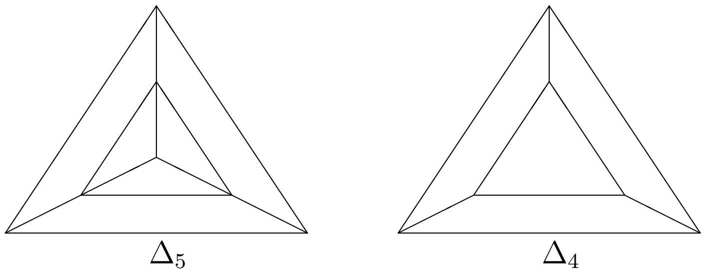


FIGURE 4

Then $f_4 : X_4 \longrightarrow Y$ is a divisorial contraction such that

(4-i) X_4 has log-terminal (not canonical) singularities,
(4-ii) $-K_{X_4}$ is f_4-ample, and
(4-iii) $\rho(X_4/Y) = 1$.

The morphism $f_5 : X_5 \longrightarrow X_4$ is a divisorial contraction with the following properties:

(5-i) K_{X_5} is f_5-ample,
(5-ii) $\rho(X_5/X_4) = 1$, and
(5-iii) X_5 has three ordinary double points.

Note that X_5 is projective over Y and $\rho(X_5/Y) = 2$.

3.5. We consider $\varphi : X := X_3 \longrightarrow W := X_5$. It is easy to check that $\mathrm{Exc}(\varphi) = \mathbb{P}^1 \amalg \mathbb{P}^1$. So, $1 \leq \rho(X/W) \leq 2$. If $\rho(X/W) = 2$, then we obtain an extremal contraction that contracts only one $\mathbb{P}^1$. We put

$$\Delta_6 = \left\{ \begin{array}{cccc} \langle e_1, e_3, e_5, e_6 \rangle, & \langle e_2, e_3, e_5, e_7 \rangle, & \langle e_4, e_5, e_6 \rangle, & \langle e_4, e_6, e_7 \rangle, \\ \langle e_4, e_5, e_7 \rangle, & \langle e_1, e_2, e_6 \rangle, & \langle e_2, e_6, e_7 \rangle, & \text{and their faces} \end{array} \right\}.$$

See the picture below (Figure 5).

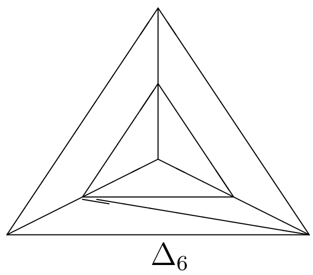


FIGURE 5

We can easily check that $X_6 := X(\Delta_6)$ is not quasi-projective. We assume that it is quasi-projective. Then there exists a strict upper convex support function h. We note that

$$\begin{aligned} e_1 + e_5 &= e_3 + e_6, \\ e_2 + e_6 &= e_1 + e_7, \\ e_3 + e_7 &= e_2 + e_5. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} h(e_1) + h(e_5) &= h(e_3) + h(e_6), \\ h(e_2) + h(e_6) &> h(e_1) + h(e_7), \\ h(e_3) + h(e_7) &= h(e_2) + h(e_5). \end{aligned}$$

This implies that

$$\sum_{i \neq 4} h(e_i) > \sum_{i \neq 4} h(e_i).$$

It is a contradiction. We checked that X_6 is not quasi-projective.

So, we do not obtain X_6 by an extremal contraction from X_3 over X_5. This is the key point of this example. Thus $\rho(X_3/X_5) = 1$.

3.6. The above arguments work without any changes if we add $-e_4$ and compactify everything. In this case, $Y = \mathbb{P}^3$, $\rho(X_3) = 4$ and $\rho(X_5) = 3$. Every variety given above becomes complete. From now on, we denote the compactified varieties with the same symbols.

3.7. We put $X = X_3$ and $W = X_5$. this flopping contraction is locally the simplest flopping contraction. We add the wall $\langle e_1, e_3 \rangle$ to Δ_3 and define it as Δ_7. More precisely, we remove the cone $\langle e_1, e_3, e_5, e_6 \rangle$ from Δ_3 and add the new cones $\langle e_1, e_3, e_5 \rangle$, $\langle e_1, e_5, e_6 \rangle$. Then $X_7 := X(\Delta_7)$ is a non-singular projective variety with $\rho(X_7) = 5$. We note that X_7 is also obtained from Y by 4-times blowing-ups with smooth centers: $X_7 \longrightarrow X_{13} \longrightarrow X_{12} \longrightarrow X_{11} \longrightarrow Y$. The next picture (Figure 6) helps us to check it.

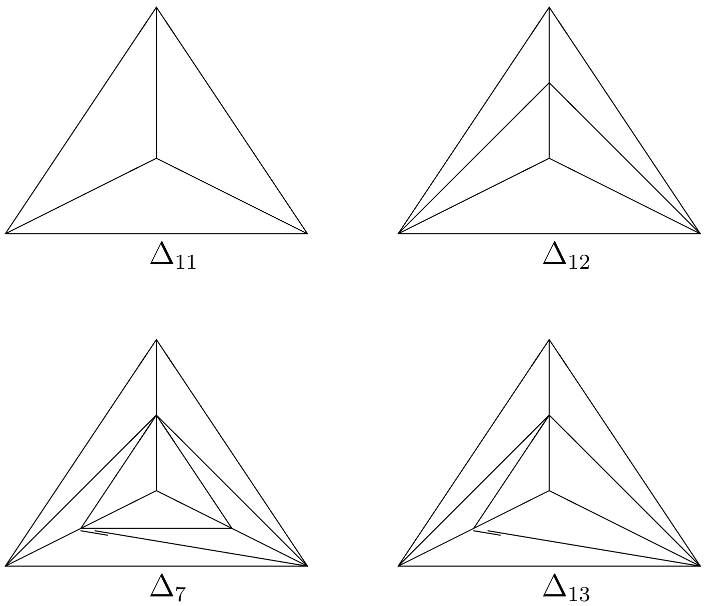


FIGURE 6

3.8. By replacing the wall $\langle e_2, e_5\rangle$ in Δ_7 with $\langle e_3, e_7\rangle$, we obtain $X_8 = X(\Delta_8)$ (see Figure 7). More precisely, we remove the cones $\langle e_2, e_3, e_5\rangle$ and $\langle e_2, e_5, e_7\rangle$ from Δ_7 and add the new cones $\langle e_2, e_3, e_7\rangle$ and $\langle e_3, e_5, e_7\rangle$.

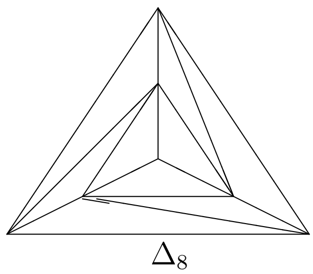
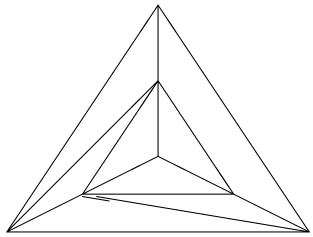


FIGURE 7

It is easy to check that X_8 is not projective (see the proof of the non-projectivity of X_6). Note that X_8 is non-singular. So, X_8 is an example of non-singular non-projective complete varieties. It is very similar to Oda's example of non-singular non-projective 3-folds (see [O, p.93 Example]).

We remove the wall $\langle e_3, e_7\rangle$ from Δ_8. This means that the cones $\langle e_2, e_3, e_7\rangle$ and $\langle e_3, e_5, e_7\rangle$ from Δ_8 and add a new cone $\langle e_2, e_3, e_5, e_7\rangle$. We put it as Δ_9 (see Figure 8).



Δ_9

FIGURE 8

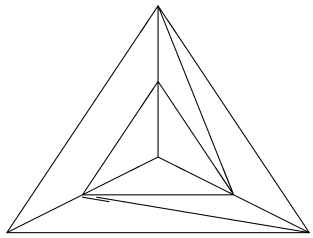
Then

$$\begin{array}{ccc} X_7 & \dashrightarrow & X_8 \\ & \searrow \quad \swarrow & \\ & X_9 & \end{array}$$

is the simplest flop. Note that X_8 is projective over X_9. However, X_8 and X_9 are not projective. This example shows that the torus invariant curve $\mathbb{P}^1 \simeq V(\langle e_2, e_5\rangle)$ on X_7 does not span any extremal rays of $NE(X_7)$ but $NE(X_7/X_9) = \mathbb{R}_{\geq 0}[V(\langle e_2, e_5\rangle)]$.

3.9. We remove the 3-dimensional cone $\langle e_1, e_3, e_5, e_6\rangle$ from Δ_3 and Δ_5. Note that we do not remove the proper faces of $\langle e_1, e_3, e_5, e_6\rangle$. Then we obtain $X \setminus P$ and $W \setminus \varphi(P)$, where P is the only one ordinary double point of X. We put $\varphi^0 : X^0 := X \setminus P \longrightarrow W^0 := W \setminus \varphi(P)$. Note that X^0 is a non-singular quasi-projective toric variety.

We claim that $\rho(X^0/W^0) = 2$. If $\rho(X^0/W^0) = 1$, then the flopping locus is $\mathbb{P}^1 \amalg \mathbb{P}^1$. It is a contradiction since the flopping locus must be irreducible when the variety is $\mathbb{Q}$-factorial (see [Fj, Theorem 3.2]). So, we obtain $\rho(X^0/W^0) = 2$. We remove the cones $\langle e_1, e_3, e_5\rangle$ and $\langle e_1, e_5, e_6\rangle$ from Δ_8 and add a new cone $\langle e_1, e_3, e_5, e_6\rangle$. We define this new fan as Δ_{10} (see Figure 9).



Δ_{10}

FIGURE 9

By flopping one $\mathbb{P}^1$ on X^0 over W^0, we obtain $X_{10}^0 := X(\Delta_{10}^0)$, where Δ_{10}^0 is $\Delta_{10} \setminus \langle e_1, e_3, e_5, e_6 \rangle$. Thus, X_{10}^0 is quasi-projective. It is easy to check that X_{10} is not projective. We put $V_1 := X_6$ and $Z := X_{10}$. So, $X_3 \dashrightarrow X_{10}$ is the desired flop in (9) in Example 2.1. It is obvious what X^+ and V_2 are. Thus, we finish the construction.

3.10. Finally, we draw a big diagram (see Figure 10).

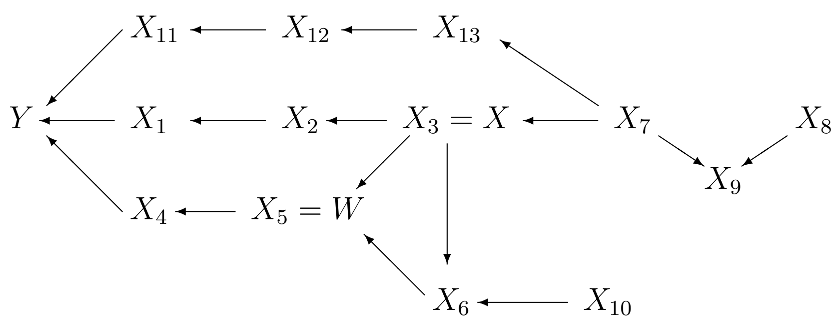


FIGURE 10

We have the following properties:

(a) $Y \simeq \mathbb{P}^3$,
(b) X_6, X_8, X_9, and X_{10} are non-projective and all the others are projective,
(c) $X_{11}, X_{12}, X_{13}, X_7$ and X_8 are non-singular,
(d) $X_3 \dashrightarrow X_{10}$ and $X_7 \dashrightarrow X_8$ are the simplest flops,
(e) $\rho(Y) = 1$, $\rho(X_1) = \rho(X_4) = \rho(X_{11}) = 2$, $\rho(X_2) = \rho(X_5) = \rho(X_{12}) = 3$, $\rho(X_3) = \rho(X_{13}) = 4$, and $\rho(X_7) = 5$.

4. Supplement

The following is a supplementary remark.

Remark 4.1. Let $f : X \longrightarrow Y$ be a toric extremal contraction, that is, f is a projective surjective toric morphism with connected fibers and $\rho(X/Y) = 1$. To investigate f, we can assume that X and Y are complete without loss of generality by [Fj, Theorems 2.10 and 2.11]. Let V be an open toric subvariety of Y and $U := f^{-1}(V)$. Assume that $\varphi := f|_U : U \longrightarrow V$ is nontrivial. If X is $\mathbb{Q}$-factorial, then $\mathrm{Pic}(X) \otimes \mathbb{Q} \longrightarrow \mathrm{Pic}(U) \otimes \mathbb{Q}$ is surjective. So, by taking the dual, we obtain that $\rho(U/V) = \rho(X/Y) = 1$. However, if X is not $\mathbb{Q}$-factorial, then $\rho(U/V)$ is not necessarily one. See Example 2.1 (8). This simple observation implies that $\mathbb{Q}$-factoriality is a very strong condition and it is difficult to describe f without $\mathbb{Q}$-factoriality.

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