CLASSIFICATION OF THREE-DIMENSIONAL TERMINAL TORIC FLIPS (PRIVATE NOTE)

OSAMU FUJINO

Abstract. We classify three-dimensional terminal toric flips.

1. Introduction

This paper is a supplement to [M, Example-Claim 14-2-5]. We classify three-dimensional terminal toric flips. The proof is a consequence of the well-known *terminal lemma* (cf. [O, §1.6]). The classification of three-dimensional flipping contractions from $\mathbb{Q}$-factorial terminal toric threefolds was stated in [KMM]. However, there is no available proof in the literature. It was claimed in [M, Example-Claim 14-2-5] that the classification was only complete with the extra assumption of the extremal rational curve passing through only one singular point. K. Matsuki (cf. [M, Remark 14-2-7 (ii)]) also stated an example, which would not fit into the classification, with the extremal rational curve passing through two singular points. Recently Y. Kawamata pointed out that Matsuki's example is not correct, having a canonical singularity which is not terminal. In this paper, we prove that the original classification is indeed complete without any extra assumption. Moreover, we classify three-dimensional flipping contractions from non-$\mathbb{Q}$-factorial terminal toric threefolds.

Acknowledgments. I would like to thank Professor Kenji Matsuki, who informed me that Professor Yujiro Kawamata pointed out an error in [M, Remark 14-2-7]. I also like to thank Doctor Hiroshi Sato for comments.

Notation. Let $v_i \in N \simeq \mathbb{Z}^3$ for $1 \le i \le k$. Then the symbol $\langle v_1, v_2, \cdots, v_k\rangle$ denotes the cone $\mathbb{R}_{\ge 0}v_1 + \mathbb{R}_{\ge 0}v_2 + \cdots + \mathbb{R}_{\ge 0}v_k$ in $N_{\mathbb{R}}$.

Date: 2005/3/1.

2000 *Mathematics Subject Classification.* Primary 14M25; Secondary 14E30.

2. Classification of three-dimensional terminal toric flips

First, we classify three-dimensional flipping contractions from $\mathbb{Q}$-factorial terminal toric threefolds. The next theorem was stated in [KMM] without proof at the end of Example 5-2-5.

Theorem 2.1 (Three-dimensional $\mathbb{Q}$-factorial terminal toric flips)**.** *Let $\varphi_R : X(\Delta) \longrightarrow Y(\Delta_Y)$ be the contraction morphism of an extremal ray R with $K_X \cdot R < 0$ of flipping type from a toric threefold with only $\mathbb{Q}$-factorial and terminal singularities. Assume that Y is affine. Then we have the following description of the flipping contraction*:

There exist two three-dimensional cones

$$\tau_4 = \langle v_1, v_2, v_3 \rangle \in \Delta,$$
$$\tau_3 = \langle v_1, v_2, v_4 \rangle \in \Delta,$$

sharing the two-dimensional wall

$$w = \langle v_1, v_2 \rangle$$

such that $[V(w)] \in R$ and such that for some $\mathbb{Z}$-coordinate of $N \simeq \mathbb{Z}^3$,

$$v_1 = (1, 0, 0), \qquad v_2 = (0, 1, 0), \qquad v_3 = (0, 0, 1),$$
$$v_4 = (a, r - a, -r),$$

or

$$v_1 = (1, 0, 0), \qquad v_2 = (0, 1, 0), \qquad v_3 = (0, 0, 1),$$
$$v_4 = (a, 1, -r),$$

where $0 < a < r$ and $\gcd(r, a) = 1$. Therefore,

$$\Delta = \{\tau_3, \tau_4, \textit{and their faces}\},$$

and

$$\Delta_Y = \{\langle v_1, v_2, v_3, v_4 \rangle, \textit{and its faces}\}.$$

Proof. By [M, Example-Claim 14-2-5], it is sufficient to prove that the (unique) rational curve that is contracted passes through only one singular point of X. Without loss of generality, we may assume that $v_1 = (1, 0, 0)$ and $v_2 = (0, 1, 0)$ since $\langle v_1, v_2 \rangle$ is a two-dimensional non-singular cone.

Seeking a contradiction, we assume that both $\langle v_1, v_2, v_3 \rangle$ and $\langle v_1, v_2, v_4 \rangle$ are singular. By the terminal lemma ([O, §1.6]), we may assume that $v_3 = (1, p, q)$ or $v_3 = (p, q - p, q)$, where $0 < p < q$ and $\gcd(p, q) = 1$. We note that $p \neq 0, q$ since $\langle v_1, v_2, v_3 \rangle$ is singular. We can write $v_4 = av_1 + bv_2 + c(k, l, -1)$ with $0 < a < c$, $0 < b < c$, $\gcd(a, c) = 1$, $\gcd(b, c) = 1$, and $k, l \in \mathbb{Z}$. We note that we assumed that $\langle v_1, v_2, v_4 \rangle$

is singular. By the terminal lemma again, at least one of $a-1$, $b-1$ and $a+b$ is divisible by c. Therefore, $a=1$, $b=1$, or $a+b=c$.

Case 1. From now on, we assume that $v_3=(1,p,q)$. In this case, v_1, v_2, v_3 are on the plane

$$x+y-\frac{p}{q}z=1.$$

Subcase 1 ($a=1$)**.** In this case, $v_4=(1+ck, b+cl, -c)$. We have

$$\frac{c}{q}v_3+v_4=(1+ck+\frac{c}{q}, b+cl+\frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$1+ck+\frac{c}{q}>0, \tag{1}$$

$$b+cl+\frac{p}{q}c>0, \tag{2}$$

and

$$1+ck+b+cl+\frac{p}{q}c<1. \tag{3}$$

The inequalities (1) and (2) follow from the condition that φ_R is small. The condition $K_X \cdot R<0$ implies the inequality (3). By (2) and (3), we have $k \le -1$. Thus

$$0<1+ck+\frac{c}{q}\le 1-c+\frac{c}{q}\le 1-\frac{1}{2}c\le 0$$

by (1). It is a contradiction.

Remark 2.2. If the reader understand Subcase 1, then he does not have to read the other subcases since the arguments are very similar.

Subcase 2 ($b=1$)**.** In this case, $v_4=(a+ck, 1+cl, -c)$. We have

$$\frac{c}{q}v_3+v_4=(a+ck+\frac{c}{q}, 1+cl+\frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$a+ck+\frac{c}{q}>0, \tag{4}$$

$$1+cl+\frac{p}{q}c>0, \tag{5}$$

and

$$a+ck+1+cl+\frac{p}{q}c<1. \tag{6}$$

By (5) and (6), $k \le -1$. So, $k = -1$ by (4). By (5), we know that $l \ge -1$. Therefore, $l = 0$ or -1 by (6).

First, we assume that $l = 0$. Then we get

$$a - c + \frac{p}{q}c < 0$$

by (6) and

$$a - c + \frac{c}{q} > 0$$

by (4). It is a contradiction.

Next, we assume that $l = -1$. Then we obtain

$$a - c + \frac{c}{q} > 0$$

by (4) and

$$1 - c + \frac{p}{q}c > 0$$

by (5). These two inequalities imply that

$$1 + a - 2c + \frac{p+1}{q}c > 0.$$

It is a contradiction.

Subcase 3 $(a + b = c)$**.** In this case, $v_4 = (a + ck, c - a + cl, -c)$. We have

$$\frac{c}{q}v_3 + v_4 = (a + ck + \frac{c}{q}, c - a + cl + \frac{p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$a + ck + \frac{c}{q} > 0, \tag{7}$$

$$c - a + cl + \frac{p}{q}c > 0, \tag{8}$$

and

$$a + ck + c - a + cl + \frac{p}{q}c < 1. \tag{9}$$

By (8) and (9), $k \le -1$. So, $k = -1$ by (7). By (8), we have $l \ge -1$. Therefore, $l = 0$ or -1 by (9).

First, we assume that $l = 0$. Then we have

$$\frac{p}{q}c < 1$$

by (9) and

$$a - c + \frac{c}{q} > 0$$

by (7). Thus,

$$1 > \frac{p}{q}c \geq \frac{c}{q} > c - a \geq 1.$$

It is a contradiction.

Next, we assume that $l = -1$. Then we obtain

$$a - c + \frac{c}{q} > 0$$

by (7) and

$$-a + \frac{p}{q}c > 0$$

by (8). By adding these two inequalities, we have

$$-c + \frac{p+1}{q}c > 0.$$

It is a contradiction.

Case 2. From now on, we assume that $v_3 = (p, q-p, q)$. In this case, v_1, v_2, v_3 are on the plane

$$x + y - \frac{q-1}{q}z = 1.$$

Subcase 4 $(a = 1)$**.** In this case, $v_4 = (1 + ck, b + cl, -c)$. We have

$$\frac{c}{q}v_3 + v_4 = (1 + ck + \frac{p}{q}c, b + cl + \frac{q-p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$1 + ck + \frac{p}{q}c > 0, \tag{10}$$

$$b + cl + \frac{q-p}{q}c > 0, \tag{11}$$

and

$$1 + ck + b + cl + \frac{q-1}{q}c < 1. \tag{12}$$

By (11) and (12), $k \leq -1$. So, $k = -1$ by (10). By (11) and (12),we know that $l = 0$ or -1.

First, we assume that $l = 0$. Then we get

$$b - c + \frac{q-1}{q}c < 0$$

by (12) and

$$1 - c + \frac{p}{q}c > 0$$

by (10). It is a contradiction.

Next, we assume that $l=-1$. Then we have

$$1-c+\frac{p}{q}c>0$$

by (10) and

$$b-c+\frac{q-p}{q}c>0$$

by (11). By adding these two inequalities, we obtain

$$1+b-c>0.$$

It is a contradiction.

Subcase 5 ($b=1$)**.** If we replace v_1 (resp. v_2) with v_2 (resp. v_1), then the same proof as in Subcase 4 works.

Subcase 6 ($a+b=c$)**.** In this case, $v_4=(a+ck, c-a+cl, -c)$. We have

$$\frac{c}{q}v_3+v_4=(a+ck+\frac{p}{q}c, c-a+cl+\frac{q-p}{q}c, 0).$$

Thus, we obtain the following three inequalities:

$$a+ck+\frac{p}{q}c>0, \tag{13}$$

$$c-a+cl+\frac{q-p}{q}c>0, \tag{14}$$

and

$$a+ck+c-a+cl+\frac{q-1}{q}c<1. \tag{15}$$

By (14) and (15), $k\leq -1$. So, $k=-1$ by (13). By (14), we have that $l\geq -1$. Thus, $l=0$ or -1 by (15).

First, we assume that $l=0$. Then we have

$$a-c+\frac{p}{q}c>0$$

by (13) and

$$\frac{q-1}{q}c<1$$

by (15). It is a contradiction since $c-a\geq 1$ and $p\leq q-1$.

Next, we assume that $l=-1$. Then we get

$$a-c+\frac{p}{q}c>0$$

by (13) and

$$-a+\frac{q-p}{q}c>0$$

by (14). By adding these two inequalities, we obtain

$$0 = a - c + \frac{p}{q}c - a + \frac{q-p}{q}c > 0.$$

It is a contradiction.

Therefore, we prove that at least one of $\langle v_1, v_2, v_3\rangle$ and $\langle v_1, v_2, v_4\rangle$ is non-singular. Thus, we have the desired description of $\varphi_R : X \longrightarrow Y$ by [M, Example-Claim 14-2-5]. □

Remark 2.3. The example in [M, Remark 14-2-7 (ii)] is not true. The cone $\langle v_1, v_2, v_3\rangle$ is not terminal. The cone $\langle v_1, v_2, v_3\rangle$ has canonical singularities.

Remark 2.4. The source space X in Theorem 2.1 is always singular.

Remark 2.5. In [M, Example-Claim 14-2-5], X is assumed to be *complete*. It is because contraction morphisms of extremal rays are constructed only for *complete* varieties in [R] and [M, Chapter 14]. For the details of non-complete toric varieties, see [FS1], [F], and [S].

Next, we classify three-dimensional flipping contractions from non-$\mathbb{Q}$-factorial terminal toric threefolds.

Theorem 2.6 (Three-dimensional non-$\mathbb{Q}$-factorial terminal toric flips)**.** *Let $\varphi_R : X(\Delta) \longrightarrow Y(\Delta_Y)$ be the contraction morphism of an extremal ray R with $K_X \cdot R < 0$ of flipping type from a toric threefold with only terminal singularities. Assume that X is not $\mathbb{Q}$-factorial and Y is affine. Then we have the following description of the flipping contraction for some $\mathbb{Z}$-coordinate of $N \simeq \mathbb{Z}^3$:*

We put

$$v_1 = (1, 0, 0), \qquad v_2 = (0, 1, 0), \qquad v_3 = (0, 0, 1),$$
$$v_5 = (-1, 1, 1), \qquad v_6 = (1, -1, 1).$$

Then, we have

$$\Delta = \{\langle v_1, v_2, v_3, v_5\rangle, \langle v_1, v_2, v_4\rangle, \textit{ and their faces}\},$$

and

$$\Delta_Y = \{\langle v_1, v_2, v_3, v_4, v_5\rangle, \textit{ and its faces}\},$$

or

$$\Delta = \{\langle v_1, v_2, v_3, v_6\rangle, \langle v_1, v_2, v_4\rangle, \textit{ and their faces}\},$$

and

$$\Delta_Y = \{\langle v_1, v_2, v_3, v_4, v_6\rangle, \textit{ and its faces}\},$$

where $v_4 = (a, r-a, -r)$ or $(a, 1, -r)$ with $0 < a < r$ and $\gcd(a, r) = 1$.

Sketch of the proof. First, we note that a non-$\mathbb{Q}$-factorial terminal toric three-dimensional singularity is an ODP.

Next, let $X' \longrightarrow X$ be a small resolution of an ODP on X. Then $\rho(X'/Y) = 2$. So, there exists another contraction $f : X' \longrightarrow Z$ over Y. It is easy to see that f is a flipping contraction such that the (unique) flipping curve passes through at most one singular point. By repeating the above argument, this singularity is not an ODP (cf. Remark 2.4). So, it is a quotient singularity. Thus, we know that the (unique) flipping curve has to pass through a quotient singularity.

Finally, we obtain the desired description from [M, Example-Claim 14-2-5] (cf. Theorem 2.1). □

Remark 2.7. In this case, it is easy to check that the flipped variety X^+ is always singular and $\rho(X^+/Y) = 2$.

Remark 2.8. In general, contraction morphisms from non-$\mathbb{Q}$-factorial toric varieties do not behave well. See examples in [F, Section 4] and [FS2].

References

[F] O. Fujino, Equivariant completions of toric contraction morphisms, preprint 2003.

[FS1] O. Fujino and H. Sato, Introduction to the toric Mori theory, Michigan Math. J. **52** (2004), 649–665.

[FS2] O. Fujino and H. Sato, An example of toric flops, preprint 2003.

[KMM] Y. Kawamata, K. Matsuda, and K. Matsuki, Introduction to the minimal model problem, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987.

[M] K. Matsuki, *Introduction to the Mori program*, Universitext. Springer-Verlag, New York, 2002. xxiv+478 pp.

[O] T. Oda, *Convex bodies and algebraic geometry. An introduction to the theory of toric varieties*, Translated from the Japanese. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], **15**. Springer-Verlag, Berlin, 1988. viii+212 pp.

[R] M. Reid, Decomposition of toric morphisms, Arithmetic and geometry, Vol. II, 395–418, Progr. Math. **36**, Birkhäuser Boston, Boston, MA, 1983.

[S] H. Sato, Combinatorial descriptions of toric extremal contractions, preprint, math.AG/0404476.

Graduate School of Mathematics, Nagoya University, Chikusa-ku Nagoya 464-8602 Japan

E-mail address: fujino@math.nagoya-u.ac.jp