CATEGORIFICATION OF CLUSTER ALGEBRAS STRUCTURES COMING FROM LIE THEORY

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The aim of this series of three lectures is to introduce the categorification of *cluster algebra* structures of coordinate rings of some unipotent cells of Lie groups. This categorification was introduced by Geiss, Leclerc and Schröer in several important articles. More precisely, our running example will consist of the following group of 4×4 matrices:

$$N := \begin{bmatrix} 1 & \mathbb{C} & \mathbb{C} & \mathbb{C} \\ 0 & 1 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & 1 & \mathbb{C} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The cluster algebra structure over $\mathbb{C}[N]$ consists of distinguished subsets of $\mathbb{C}[N]$ called *clusters* and a *mutation operation* which permits to reach other clusters in an inductive way. More precisely, if $\mathbf{x} := \{x_1, x_2, \ldots, x_n\}$ is a cluster, its mutation in direction k is a cluster obtained by a replacement $\mathbf{x} \setminus \{x_k\} \cup \{x'_k\}$ where a relation $x_k x'_k = M_1 + M_2$ is satisfied (M_1 and M_2 are monomials in the variables of $\mathbf{x} \setminus \{x_k\}$).

The categorification is then achieved using the category $\operatorname{mod} \Pi$ of finitely generated module over the preprojective algebra of type A_3. More precisely, clusters correspond to special objects of $\operatorname{mod} \Pi$ called *cluster tilting* and there is a mutation of cluster tilting objects which consist of exchanging exactly one direct summand of these objects. Thus, a cluster character $\varphi : \operatorname{mod} \Pi \to \mathbb{C}[N]$ permits to recover the clusters of the coordinate ring $\mathbb{C}[N]$.

In the first lecture, we will introduce the definition of a cluster algebra and the cluster structure of $\mathbb{C}[N]$, illustrating by the problem of total positive matrices (*i.e.* matrices in N all the non-trivial minor of which are positive). We will also introduce the category $\operatorname{mod} \Pi$.

In the second lecture, we will focus on the categorical setting: the cluster tilting objects, their mutations, and the effect of these mutations on their endomorphism rings.

In the third lecture, we will introduce the cluster character which permits to make the link with the first lecture. If the time permits it, we will also speak about some generalizations.

Detailed notes of an extended version of these lectures will be available on:

`http://www.math.nagoya-u.ac.jp/~demonet/tokyo/clusters.pdf`