EXAMPLE OF CATEGORIFICATION OF A CLUSTER ALGEBRA

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Abstract. We present here two detailed examples of additive categorifications of the cluster algebra structure of a coordinate ring of a maximal unipotent subgroup of a simple Lie group. The first one is of simply-laced type \((A_3)\) and relies on an article by Geiß, Leclerc and Schröer. The second is of non simply-laced type \((C_2)\) and relies on an article by the author of this note. This is aimed to be accessible, specially for people who are not familiar with this subject.

1. Introduction: the total positivity problem

Let \(N\) be the subgroup of \(\text{SL}_4(\mathbb{C})\) consisting of upper triangular matrices with diagonal 1. We say that \(X \in N\) is totally positive if its 12 non-trivial minors are positive real numbers (a minor is non-trivial if it is not constant on \(N\) and not product of other minors). As a consequence of various results of Fomin and Zelevinsky [3] (see also [1]), in a (very) special case, we get

**Proposition 1** (Fomin-Zelevinsky). \(X \in N\) is totally positive if and only if the minors \(\Delta_{14}(X), \Delta_{24}(X), \Delta_{123}(X), \Delta_{234}(X), \Delta_2(X), \Delta_3(X)\) are positive.

where \(\Delta_{\ell_1\ldots\ell_k}(X)\) is the minor of \(X\) with rows \(\ell_1, \ldots, \ell_k\) and columns \(c_1, \ldots, c_k\).

Remark that, as the algebraic variety \(N\) has dimension 6, we can not expect to find a criterion with less than 6 inequalities to check the total positivity of a matrix.

To prove this, just remark that we have the following equality:

\[
\Delta_{24}\Delta_{34} = \Delta_{234}\Delta_4^2 + \Delta_2^3 \Delta_{34}^2
\]

which immediately implies that \(\Delta_{14}(X), \Delta_{24}(X), \Delta_{123}(X), \Delta_{234}(X), \Delta_2(X), \Delta_3(X)\) are positive if and only if \(\Delta_1(X), \Delta_{23}(X), \Delta_{234}(X), \Delta_{234}(X), \Delta_2(X), \Delta_3(X)\) are positive. Such an equality is called an exchange identity. In Figure 1, we wrote 14 sets of minors which are related by exchange identities whenever they are linked by an edge. As every minor appears in this graph, it induces the previous proposition.

These observations lead to the definition of a cluster algebra [4]. A cluster algebra is an algebra endowed with an additional combinatorial structure. Namely, a (generally infinite) set of distinguished elements called cluster variables grouped into subsets of the same cardinality \(n\), called clusters and a finite set \(\{x_{n+1}, x_{n+2}, \ldots, x_m\}\) called the set of coefficients. For each cluster \(\{x_1, x_2, \ldots, x_n\}\), the extended cluster \(\{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}\) is a transcendence basis of the algebra. Moreover, each cluster \(\{x_1, x_2, \ldots, x_n\}\) has \(n\)

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neighbours obtained by replacing one of its elements $x_k$ by a new one $x'_k$ related by a relation

$$x_k x'_k = M_1 + M_2$$

where $M_1$ and $M_2$ are mutually prime monomials in \{x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m\}, given by precise combinatorial rules. These replacements, called mutations and denoted by $\mu_k$ are involutive. For precise definitions and details about these constructions, we refer to [4].

In the previous example, the coefficients are $\Delta_1^1$, $\Delta_{12}^{12}$ and $\Delta_{123}^{123}$ and the cluster variables are all the other non-trivial minors. The extended clusters are the sets appearing at the vertices of Figure 1.
The aim of the following sections is to describe examples of additive categorifications of cluster algebras. It consists of enhancing the cluster algebra structure with an additive category, some objects of which reflect the combinatorial structure of the cluster algebra; moreover, there is an explicit formula, the cluster character associating to these particular objects elements of the algebra, in a way which is compatible with the combinatorial structure. The examples we develop here rely on (abelian) module categories. They are particular cases of categorifications by exact categories appearing in [6] (simply-laced case) and [2] (non simply-laced case). The study of cluster algebras and their categorifications has been particularly successful these last years. For a survey on categorification by triangulated categories and a much more complete bibliography, see [7].

2. THE PREPROJECTIVE ALGEBRA AND THE CLUSTER CHARACTER

Let $Q$ be the following quiver (oriented graph):

$$
\begin{array}{c}
1 \\
\alpha \\
\alpha^* \\
2 \\
\beta \\
\beta^* \\
3
\end{array}
$$

As usual, denote by $\mathbb{C}Q$ the $\mathbb{C}$-algebra, a basis of which is formed by the paths (including 0-length paths supported by each of the three vertices) and the multiplication of which is defined by concatenation of paths when it is possible and vanishes when paths can not be composed (we write here the composition from left to right, on the contrary to the usual composition of maps). Thus, a (right) $\mathbb{C}Q$-module is naturally graded by idempotents (0-length paths) corresponding to vertices and the action of arrows seen as elements of the algebra can naturally be identified with linear maps between the corresponding homogeneous subspaces of the representation. We shall use the following right-hand side convenient notation:

$$
\begin{array}{c}
\mathbb{C} \\
\mathbb{C}^2 \\
\mathbb{C}^2 \\
1 \\
0 \\
1 \\
0
\end{array}
\begin{array}{c}
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 1
\end{pmatrix}
\end{array}
= 
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
$$

where each of the digits represents a basis vector of the representation and each arrow a non-zero scalar (1 when not specified) in the corresponding matrix entry.

Let us now introduce the preprojective algebra of $Q$:

**Definition 2.** The preprojective algebra of $Q$ is defined by

$$
\Pi_Q = \frac{\mathbb{C}Q}{(\alpha\alpha^*, \alpha^*\alpha + \beta^*\beta, \beta\beta^*)}
$$

the representations of which are seen as particular representations of $\mathbb{C}Q$ (in other words, $\text{mod} \Pi_Q$ is a full subcategory of $\text{mod} \mathbb{C}Q$).
Example 3. Among the following representations of $CQ$, the first one and the second one are representations of $Q$:

\[ \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
  1 \\
  \circ \\
  2 \\
  \circ \\
  3 \\
  \circ
\end{array}
\end{array}
\end{array}
\quad ; \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
  1 \\
  \circ \\
  2 \\
  \circ \\
 -1 \\
  \circ
\end{array}
\end{array}
\end{array}
\quad ; \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
  1 \\
  \circ \\
  2 \\
  \circ \\
  3 \\
  \circ
\end{array}
\end{array}
\end{array}
\quad ; \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
  1 \\
  \circ \\
  2 \\
  \circ \\
 -1 \\
  \circ
\end{array}
\end{array}
\end{array}
\end{array} \]

One of the property, which is discussed in many places (for example in [6]), of the preprojective algebra of $Q$, fundamental for this categorification, is

**Proposition 4.** The category $\text{mod} \, \Pi_Q$ is stably 2-Calabi-Yau. In other words, for every $X, Y \in \text{mod} \, \Pi_Q$,

\[ \text{Ext}^1(Y, X) \cong \text{Ext}^1(Y, X)^* \]

functorially in $X$ and $Y$, where $\text{Ext}^1(Y, X)^*$ is the $\mathbb{C}$-dual of $\text{Ext}^1(Y, X)$. In particular, it is a Frobenius category (is has enough projective objects and enough injective objects and they coincide).

Let us now define the three following one-parameter subgroups of $N$:

\[
x_1(t) = \begin{pmatrix}
1 & t & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad x_2(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & t & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad x_3(t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

For $X \in \text{mod} \, \Pi_Q$ and any sequence of vertices $a_1, a_2, \ldots, a_n$ of $Q$, we denote by

\[ \Phi_{X, a_1a_2\ldots a_n} = \left\{ 0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X \mid \forall i \in \{1, 2, \ldots, n\}, \frac{X_i}{X_{i-1}} \cong S_{a_i} \right\} \]

the variety of composition series of $X$ of type $a_1a_2\ldots a_n$ ($S_{a_i}$ is the simple module, of dimension 1, supported at vertex $a_i$). This is a closed algebraic subvariety of the product of Grassmannians

\[ \text{Gr}_1(X) \times \text{Gr}_2(X) \times \cdots \times \text{Gr}_n(X). \]

We denote by $\chi$ the Euler characteristic. Using results of Lusztig and Kashiwara-Saito, Geiß-Leclerc-Schröer proved the following result:

**Theorem 5 ([6]).** Let $X \in \text{mod} \, \Pi_Q$. There is a unique $\varphi_X \in \mathbb{C}[N]$ such that

\[ \varphi_X(x_{a_1(t_1)}x_{a_2(t_2)} \cdots x_{a_6(t_6)}) = \sum_{i_1, i_2, \ldots, i_6 \in \mathbb{N}} \chi \left( \Phi_{X, a_1^{i_1}a_2^{i_2}\ldots a_6^{i_6}} \right) \frac{t_1^{i_1}t_2^{i_2}\cdots t_6^{i_6}}{i_1!i_2!\cdots i_6!} \]

for every word $a_1a_2a_3a_4a_5a_6$ representing the longest element of $S_4$ ($a_k^{i_k}$ is the repetition $i_k$ times of $a_k$).

The map $\varphi : \text{mod} \, \Pi_Q \to \mathbb{C}[N]$ is called a cluster character.

**Remark 6.** (1) The uniqueness in the previous theorem is easy because it is well known that

\[ x_{a_1(t_1)}x_{a_2(t_2)} \cdots x_{a_6(t_6)} \]

runs over a dense subset of $N$ ;
The existence is much harder and strongly relies on the construction of semicanonical bases by Lusztig [8]. In particular, the fact that it does not depend on the choice of $a_1 a_2 a_3 a_4 a_5 a_6$ is not clear a priori (see the following examples).

**Example 7.** We suppose that $a_1 a_2 a_3 a_4 a_5 a_6 = 213213$. Then

$$x_{a_1} (t_1) x_{a_2} (t_2) x_{a_3} (t_3) x_{a_4} (t_4) x_{a_5} (t_5) x_{a_6} (t_6) = \begin{pmatrix} 1 & t_2 + t_5 & t_2 t_4 & t_2 t_4 t_6 \\ 0 & 1 & t_1 + t_4 & t_1 t_3 + t_1 t_6 + t_4 t_6 \\ 0 & 0 & 1 & t_3 + t_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$  

- The module $S_1$ has only one composition series, of type 1. Therefore $\Phi_a (S_1)$ is one point and $\Phi_a (S_1) = \emptyset$ for any other $a$. Identifying the two members in the formula of the previous theorem,

$$\varphi_{S_1} (x_{a_1} (t_1) x_{a_2} (t_2) x_{a_3} (t_3) x_{a_4} (t_4) x_{a_5} (t_5) x_{a_6} (t_6)) = t_2 + t_5 = \Delta_2^1.$$  

- The module

$$P_2 = 1 \leftarrow 2 \rightarrow 3$$

has two composition series, of type 2312 and 2132. Therefore,

$$\varphi_{P_2} (x_{a_1} (t_1) x_{a_2} (t_2) x_{a_3} (t_3) x_{a_4} (t_4) x_{a_5} (t_5) x_{a_6} (t_6)) = t_1 t_2 t_3 t_4 = \Delta_{34}^{12}.$$  

Remark that, in this case, the only composition series which is playing a role is 2132, even if the situation is symmetric. This justify the second part of the previous remark.

The other indecomposable representations of $\Pi_Q$ and their cluster character values are collected in Figure 2.

Two important properties of this cluster character were proved by Geiß-Leclerc-Schröer (see for example [6]):

**Proposition 8.** Let $X, Y \in \text{mod} \Pi_Q$.

1. $\varphi_{X \oplus Y} = \varphi_X \varphi_Y$.  

<table>
<thead>
<tr>
<th>$X \in \text{mod} \Pi_Q$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$1 \rightleftarrows 2$</th>
<th>$1 \rightleftarrows 3$</th>
<th>$2 \rightleftarrows 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi_X \in \mathbb{C}[N]$</td>
<td>$\Delta_2^1$</td>
<td>$\Delta_3^2$</td>
<td>$\Delta_4^3$</td>
<td>$\Delta_{23}^{12}$</td>
<td>$\Delta_5^4$</td>
<td>$\Delta_{34}^{23}$</td>
</tr>
</tbody>
</table>

**Figure 2. Cluster character**
(2) Suppose that \( \dim \text{Ext}^1(X, Y) = 1 \) (and therefore \( \dim \text{Ext}^1(Y, X) = 1 \)) and let
\[
0 \to X \to T_a \to Y \to 0 \quad \text{and} \quad 0 \to Y \to T_b \to X \to 0
\]
be two (unique up to isomorphism) non-split short exact sequences. Then
\[
\varphi_X \varphi_Y = \varphi_{T_a} + \varphi_{T_b}.
\]

3. Minimal approximations

This section recall the definition and elementary properties of approximations. It is there for the sake of ease. In what follows, \( \text{mod} \Pi_Q \) can be replaced by any additive Hom-finite category over a field.

**Definition 9.** Let \( X \) and \( T \) be two objects of \( \text{mod} \Pi_Q \). A \textit{left add}(\( T \))-approximation of \( X \) is a morphism \( f : X \to T' \) such that

- \( T' \in \text{add}(T) \) (which means that every indecomposable summand of \( T' \) is an indecomposable summand of \( T \)) ;
- every morphism \( g : X \to T \) factors through \( f \).

If, moreover, there is no strict direct summand \( T'' \) of \( T' \) and left \textit{add}(\( T \))-approximation \( f' : X \to T'' \), then \( f \) is said to be a \textit{minimal left add}(\( T \))-approximation.

In the same way, we can define

**Definition 10.** Let \( X \) and \( T \) be two objects in \( \text{mod} \Pi_Q \). A \textit{right add}(\( T \))-approximation of \( X \) is a morphism \( f : T' \to X \) such that

- \( T' \in \text{add}(T) \);
- every morphism \( g : T \to X \) factors through \( f \).

If, moreover, there is no strict direct summand \( T'' \) of \( T' \) and right \textit{add}(\( T \))-approximation \( f' : T'' \to X \), then \( f \) is said to be a \textit{minimal right add}(\( T \))-approximation.

Now, a classical proposition which permits to explicitly compute approximations:

**Proposition 11.** Let \( X \) and \( T \simeq T_1^i \oplus T_2^j \oplus \cdots \oplus T_n^m \) be two objects in \( \text{mod} \Pi_Q \) (the \( T_i \)'s are non-isomorphic indecomposable). For \( i, j \in \{1, \ldots, n\} \), we denote by \( I_{ij} \) the subvector space of \( \text{Hom}(T_i, T_j) \) consisting of the non-invertible morphisms \( (I_{ij} = \text{Hom}(T_i, T_j) \) if \( i \neq j \)). Thus, for \( j \in \{1, \ldots, n\} \), we obtain a linear map

\[
\bigoplus_{i \in \{1, \ldots, n\}} I_{ij} \otimes \text{Hom}(X, T_i) \xrightarrow{\varphi_j} \text{Hom}(X, T_j)
\]

\[
(g, f) \mapsto g \circ f.
\]

Let \( B_j \) be a basis of \( \text{coker} \varphi_j \) lifted to \( \text{Hom}(X, T_j) \). Then the morphism
\[
X \xrightarrow{(f)_{j \in \{1, \ldots, n\}, f \in B_j}} \bigoplus_{j \in \{1, \ldots, n\}} T_j^{|B_j} \,
\]

is a minimal left \textit{add}(\( T \))-approximation of \( X \). Moreover, any minimal left \textit{add}(\( T \))-approximation of \( X \) is isomorphic to it.
The previous proposition has a dual version which permits to compute minimal right approximations. In practice, this computation relies on searching morphisms up to factorization through other objects. There is an explicit example of computation in Example 19.

### 4. Maximal rigid objects and their mutations

Let us introduce the objects the combinatorics of which will play the role of the cluster algebra structure.

**Definition 12.** Let \( X \in \text{mod} \, \Pi_Q \).

- The module \( X \) is said to be **rigid** if it has no self-extension, (i.e., \( \text{Ext}^1(X, X) = 0 \)).
- The module \( X \) is said to be **basic maximal rigid** if it is basic (i.e., it does not have two isomorphic indecomposable summands), rigid, and maximal for these two properties.

**Remark 13.** A basic maximal rigid \( \Pi_Q \)-module contains \( \Pi_Q \) as a direct summand (because \( \Pi_Q \) is both projective and injective and therefore has no extension with any module).

**Example 14.** The object

\[
\begin{array}{cccccc}
1 & & & & 3 & \\
2 & & & \oplus & 3 & \\
2 & & \oplus & 1 & 2 & \\
1 & \oplus & 2 & \oplus & 3 & \\
1 & 2 & 3 & 2 & 1 & 3 & 2 & 1 & 3 \\
\end{array}
\]

the last three summands of which are the indecomposable projective-injective \( \Pi_Q \)-modules, is basic maximal rigid. It is easy to check that it is basic and rigid, but more difficult to prove that it is maximal for these properties (see [6] for more details).

**Remark 15.** We can prove that all basic maximal rigid objects have the same number of indecomposable summands (six in the example we are talking about).

The following result permits to define a mutation on basic maximal rigid objects. Considered as an operation on isomorphism classes of basic maximal rigid objects, the induced combinatorial structure will correspond to the one of a cluster algebra.

**Theorem 16** ([6]). Let \( T \simeq T_1 \oplus T_2 \oplus T_3 \oplus P_1 \oplus P_2 \oplus P_3 \in \text{mod} \, \Pi_Q \) be basic maximal rigid such that \( P_1, P_2 \) and \( P_3 \) are the indecomposable projective \( \Pi_Q \)-modules and \( T_1, T_2 \) and \( T_3 \) are indecomposable non-projective \( \Pi_Q \)-modules. Then, for \( i \in \{1, 2, 3\} \), there are two (unique) short exact sequences

\[
0 \rightarrow T_i \xrightarrow{f} T_a \xrightarrow{f'} T_i^* \rightarrow 0 \quad \text{and} \quad 0 \rightarrow T_i^* \xrightarrow{g} T_b \xrightarrow{g'} T_i \rightarrow 0
\]

such that

1. \( f \) and \( g \) are minimal left \( \text{add}(T/T_i) \)-approximations ;
2. \( f' \) and \( g' \) are minimal right \( \text{add}(T/T_i) \)-approximations ;
3. \( T_i^* \) is indecomposable and non-projective ;
4. \( \dim \text{Ext}^1(T_i, T_i^*) = \dim \text{Ext}^1(T_i^*, T_i) = 1 \) and the two short exact sequences do not split ;
5. \( \mu_i(T) = T/T_i \oplus T_i^* \) is basic maximal rigid ;
(6) $T_a$ and $T_b$ do not have common summands.

Remark 17. In the previous theorem, the existence and uniqueness, regarding the first two conditions, are automatic, except the fact that the extremities of the two short exact sequences coincide up to order. This fact strongly relies on the stably 2-Calabi-Yau property. It implies that $\mu_i$ is involutive.

Definition 18. In the previous theorem, $\mu_i$ is called the mutation in direction $i$. The short exact sequences appearing are called exchange sequences.

Example 19. Let

$$ T = \begin{array}{c}
1 & 2 & 3 \\
\oplus & 2 & 3 \\
\oplus & 1 & 2 \\
& 3 & 1 \\
\end{array} $${8

Using Proposition 11, we get a left add $\left(T/\begin{array}{c}
2 & 3 \\
\end{array}\right)$-approximation of $\begin{array}{c}
2 & 3 \\
\end{array}$:

$$ \begin{array}{c}
0 \\
2 & 3 \\
\rightarrow & 1 & 2 & 3 \\
\end{array} $$

and computing the cokernel, we get the exchange sequence:

$$ 0 \rightarrow 2 & 3 \rightarrow 1 & 2 & 3 \rightarrow S_1 \rightarrow 0 $$

so that

$$ \mu_2(T) = \begin{array}{c}
1 & 2 & 3 \\
\oplus & S_1 & \oplus & 1 & 2 & 3 \\
\oplus & 1 & 2 & 3 & \oplus & 2 & 3 \\
& 2 & 1 & 3 & \oplus & 1 & 2 \\
\end{array} $$

Doing mutation in the reverse direction:

$$ 0 \rightarrow S_1 \rightarrow 2 & 3 \rightarrow 2 & 3 \rightarrow 0. $$

Let us now compute $\mu_1\mu_2(T)$ with its two exchange sequences:

$$ 0 \rightarrow 1 & 2 & 3 \rightarrow S_1 \oplus 1 & 2 & 3 \rightarrow 1 & 2 \rightarrow 0 $$

$$ 0 \rightarrow 1 & 2 \rightarrow 1 & 2 \oplus 1 & 2 \rightarrow 1 & 2 & 3 \rightarrow 0 $$

$$ \mu_1\mu_2(T) = \begin{array}{c}
2 & 1 & 2 \oplus & S_1 & \oplus & 1 & 2 & 3 \\
\oplus & 1 & 2 & 3 & \oplus & 2 & 3 & \oplus & 1 & 2 \\
\end{array} $$
Computing inductively all the mutations, we obtain the exchange graph of maximal rigid objects of $\Pi Q$ (Figure 3).

Then, using Proposition 8 and Theorem 16 together with other technical results, we get the following proposition:

**Proposition 20** ([6]). *If we project the mutation of maximal rigid objects to $\mathbb{C}[N]$ through the cluster character $\varphi$, we get a cluster algebra structure on $\mathbb{C}[N]$ (in the sense of [4]). Moreover, this structure is the one proposed combinatorially in [1]. Under this projection, we get the correspondence:*

\[
\begin{align*}
\{\text{non projective indecomposable objects}\} & \leftrightarrow \{\text{cluster variables}\} \\
\{\text{projective indecomposable objects}\} & \leftrightarrow \{\text{coefficients}\} \\
\{\text{basic maximal rigid objects}\} & \leftrightarrow \{\text{extended clusters}\}
\end{align*}
\]

**Example 21.** Taking the notation of Example 19 and looking at Figure 2, we get:

\[
\Delta_2^1 \Delta_4^1 = \varphi S_1 \varphi,
\]

and

\[
\Delta_2^1 \Delta_3^1 = \varphi S_1 \varphi + \varphi S_1 \varphi = \Delta_2^1 \Delta_3^1 + \Delta_2^1 \Delta_3^1.
\]

which can be easily checked by hand. These are part of the equalities which appear in the proof of Proposition 1.

5. FROM SIMPLY-LACED CASE TO GENERAL ONE

Define the following symplectic form:

\[
\Psi = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

and the subgroup

\[
N' = \{M \in N|M\Psi M = \Psi\} \quad \text{or, equivalently} \quad N' = N_{2Z}/2Z
\]

where $Z/2Z = \langle g \rangle$ acts on $N$ by $M \mapsto \Psi^{-1}(gM^{-1})\Psi$. The group $N'$ is a maximal unipotent subgroup of a symplectic group of type $C_2$. 
The only non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on $Q$ induces an action on $\Pi_Q$ and therefore on $\text{mod} \Pi_Q$. Denote by $\pi : \mathbb{C}[N] \rightarrow \mathbb{C}[N']$ the canonical projection. We can now formulate the following result:

**Theorem 22** ([2]).

1. If $T$ is a $\mathbb{Z}/2\mathbb{Z}$-stable basic maximal rigid $\Pi_Q$-module, then $\mu_1 \mu_3(T) = \mu_3 \mu_1(T)$. Moreover, $\mu_1 \mu_3(T)$ and $\mu_2(T)$ are also $\mathbb{Z}/2\mathbb{Z}$-stable.
2. If $X \in \text{mod} \Pi_Q$, then $\pi(\varphi_X) = \pi(\varphi_{\bar{g}X})$.
3. If we denote $\bar{\mu}_2 = \mu_2$ and $\bar{\mu}_1 = \mu_1 \mu_3 = \mu_3 \mu_1$, acting on the set of $\mathbb{Z}/2\mathbb{Z}$-stable maximal rigid $\Pi_Q$-modules, $\bar{\mu}$ induces through $\pi \circ \varphi$ the structure of a cluster algebra on $\mathbb{C}[N']$, the clusters of which are projections of the $\mathbb{Z}/2\mathbb{Z}$-stable ones of $\mathbb{C}[N]$. 
Example 23. We have
\[
\Delta_{12} = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = a_{12}a_{23} - a_{13}
\]
and
\[
\Delta_{23} = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} = a_{24}.
\]
Moreover,
\[
\psi^{-1} \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \psi
\]
\[
= \begin{pmatrix} 1 & a_{34} & a_{23}a_{34} - a_{24} & a_{12}a_{23}a_{34} - a_{12}a_{24} - a_{13}a_{34} + a_{14} \\ 0 & 1 & a_{23} & a_{12}a_{23} - a_{13} \\ 0 & 0 & 1 & a_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
which implies that, as expected,
\[
\pi(\Delta_{12}) = \pi\varphi_1 = \pi\varphi_2 = \pi\varphi_3 = \pi(\Delta_{23}).
\]
The exchange graph of the \(\mathbb{Z}/2\mathbb{Z}\)-stable basic maximal rigid objects of mod \(\Pi_Q\) is presented on Figure 4, in relation to the exchange graph of the basic maximal rigid objects. It permits, in view of Figure 1 to describe the clusters of \(\mathbb{C}[\mathbb{N}']\):

\[
\Delta_1 = \Delta_{1234}, \Delta_{34}, \\
\Delta_{12} = \Delta_{1234}, \Delta_{34}, \\
\Delta_{13} = \Delta_{1234}, \Delta_{34},
\]

6. Scope of these results and consequences
The example presented here can be generalized to the coordinate rings of:

- The groups of the form
  \[
  N(w) = N \cap (w^{-1}Nw) \quad \text{and} \quad N^w = N \cap (BwB^{-1})
  \]
where $N$ is a maximal unipotent subgroup of a Kac-Moody group, $N_-$ its opposite unipotent group, $B_-$ the corresponding Borel subgroup, and $w$ is an element of the corresponding Weyl group. In particular, if $N$ is of Lie type and $w$ is the longest element, then $N(w) = N$.

- Partial flag varieties corresponding to classical Lie groups.

These results were obtained in [5] and [6] for the simply-laced cases and in [2] for the non simply-laced cases.

It permits for example to prove in these cases that all the cluster monomials (products of elements of a same extended cluster) are linearly independent (result which is now generalized but was new at that time) and other more specific results (for example the
classification of partial flag varieties the coordinate rings of which have finite cluster type, that is a finite number of clusters).

REFERENCES


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