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Combinatorics of Mutations in Representation Theory

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Contents

Introduction

My recent research topics are representation theory and categorification of cluster algebras. Given an algebra A over a field $\Bbbk$, it is natural to study the category $\operatorname{mod} A$ of its representations (maybe with some restrictions). A very fruitful idea in this domain is to associate to $\operatorname{mod} A$ certain combinatorial invariants. For example:

- The *Auslander-Reiten quiver of* $\operatorname{mod} A$ is the oriented graph, the vertices of which are the isomorphism classes of indecomposable representations, and arrows from X to Y form a basis of irreducible morphisms. In good cases, in particular if the algebra is finite dimensional over $\Bbbk$, this graph is endowed with a partial automorphism, the *Auslander-Reiten translation* τ and the knowledge of this graph permits to study $\operatorname{mod} A$, in particular from a homological point of view.
- The *exchange graph* of a certain class of representations $\mathscr{O}$ (often a variation of so-called *maximal rigid* representations). Under certain hypotheses and for good classes of objects, we have a *mutation operation* between isomorphism classes of objects $\mathscr{O}$ which permits, given a representation $T \in \mathscr{O}$, decomposed into direct summands $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$, to obtain, in a unique way for each $i \in \{1, 2, \ldots, n\}$, a new object

  $$\mu_i(T) := T_1 \oplus T_2 \oplus \cdots \oplus T_{i-1} \oplus T_i^* \oplus T_{i+1} \oplus \cdots \oplus T_n$$

  with same direct summands as T, except T_i which is exchanged with T_i^*. Most of the time, T_i^* is uniquely determined by the existence of a short exact sequence $0 \to T_i \to \overline{T} \to T_i^* \to 0$ with strong properties. The exchange graph of $\mathscr{O}$ is the (non-oriented) graph, the vertices of which are isomorphism classes of representations in $\mathscr{O}$ and having arcs linking representations related by a mutation.

Natural arising questions are:

- classification questions: can we classify, for a certain class of algebras, these combinatorial invariants, or at least obtain certain properties of them?
- inverse questions: from certain properties of an invariant, what can be deduced about the algebra or its category of representations?
- comparison questions: if two algebras are related in a certain way (for example, one is a subalgebra of the other, or a quotient), can we deduce some relations between their invariants?

Another line of questioning, less natural at first glance, consists in interpreting certain structure of combinatorial nature as coming from a certain category of representations of an algebra. For example, for a certain oriented graph, is there an algebra such that this graph is its Auslander-Reiten quiver or a subgraph of it? These techniques, called *categorification* has been very fruitful during recent decades. Let us mention in particular the case of categorification of cluster algebras. Cluster algebras are algebras endowed with an exchange graph (see Section 4). The categorification of a cluster algebra $\mathcal{A}$ consists, roughly, to associate to it another algebra Λ and a class $\mathscr{O}$ of representations of Λ with a mutation, in such a way that the exchange graph of $\mathscr{O}$ and the exchange graph of $\mathcal{A}$ are naturally isomorphic. The power of these methods, combining algebra and combinatorics, comes from the fact that they permit to use in the same context ideas of different nature (for example counting on the one hand and homological algebra on the other hand). Remark that the word "categorification" has been used in several different contexts with different meanings. We use it here in a very broad sense to describe a strategy to attack certain problems rather than a precise concept. The precise concept of *additive categorifications of cluster algebras* is explained in Section 4.

Notice, before discussing more precisely certain questions, that it is reasonable, for certain algebras, to restrict the category of representations that we study. This is for example the case for orders on discrete valuation rings. If R is a discrete valuation ring (for example $R = \Bbbk[\![t]\!]$), an order Λ over R is an R-algebra which is, as a representation of R, free and of finite rank (*i.e.*

isomorphic to R^n for a certain n). An example of order on $R = \mathbb{C}[\![t]\!]$ is the matrix algebra

$$\Lambda := \begin{bmatrix} R & (t) \\ (t) & R \end{bmatrix}$$

which, as an R-module, is isomorphic to R^4 ((t) is the ideal generated by t). The category of representations of Λ is generally too big to be studied in a satisfactory way. Nevertheless, the category of its Cohen-Macaulay modules

$$\mathsf{CM}\,\Lambda = \{M \in \operatorname{mod}\Lambda \mid M \text{ is free and of finite rank over } R\}$$

is a natural, more accessible, variant in this case. In the previous example, the indecomposable objects of $\mathsf{CM}\,\Lambda$ can be written, up to isomorphism, as row matrices in the following way:

$$\begin{bmatrix} R & (t) \end{bmatrix} \quad ; \quad \begin{bmatrix} (t) & R \end{bmatrix} \quad ; \quad \begin{bmatrix} R & R \end{bmatrix}$$

where Λ acts by right multiplication. Moreover, $\mathsf{CM}\,\Lambda$ has an Auslander-Reiten quiver, described in Example 2.6(c).

This document is a survey about some results I obtained in collaboration with several colleagues during last few years. Sections 1 and 2 introduce fundamental concepts of my research, while Sections 3 to 6 present results of my research. I also present at the end of each of these sections some open problems I plan to study in the future. I finish the introduction by describing quickly the content of Sections 3 to 6.

Lattices of torsion classes and τ-tilting modules. We consider a finite dimensional algebra A over a field $\Bbbk$. In [AIR], Adachi–Iyama–Reiten have introduced *τ-tilting A-modules*, completing the class of tilting modules from the point of view of mutations. An A-module X is *τ-rigid* if $\mathsf{Hom}_A(X, \tau X) = 0$ where τ is the Auslander-Reiten translation. It implies that X is rigid, *i.e.* $\mathsf{Ext}^1_A(X, X) = 0$. A *$\tau$-tilting pair* (X, P) consists of $X \in \operatorname{mod} A$ basic and τ-rigid, $P \in \operatorname{mod} A$ basic and projective such that $\mathsf{Hom}_A(P, X) = 0$ and (X, P) is maximal for these properties. Equivalently to the maximality, the number of indecomposable direct summands of $X \oplus P$ is the number of isomorphism classes of indecomposable projective A-modules.

It turns out that the set $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ of isomorphism classes of τ-tilting pairs is endowed with an oriented mutation. In other terms, for $(X, P) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$ and an indecomposable summand $U = (X_0, 0)$ or $U = (0, P_0)$ of (X, P), there is exactly one indecomposable pair V different from U satisfying that $(X, P)/U \oplus V$ is a τ-tilting pair. Moreover we can define a direction of mutation between the two objects (X, P) and $(X, P)/U \oplus V$. Then, the oriented *exchange graph* Γ_A of τ-tilting pairs has set of vertices $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ and an arrow from (X, P) to (Y, Q) if there is a mutation going from (X, P) to (Y, Q). It has one source, namely $(A, 0)$ and one sink, $(0, A)$. We refer to Section 2 for more details about this theory.

For instance, consider the matrix algebra

$$A = \begin{bmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{bmatrix},$$

or, in other terms, the path algebra of the quiver $1 \to 2$. The exchange graph Γ_A is

$$\begin{tikzcd}
 & \left(\begin{bmatrix} \Bbbk & \Bbbk \end{bmatrix} \oplus \begin{bmatrix} \Bbbk & 0 \end{bmatrix}, 0\right) \arrow[rr] & & \left(\begin{bmatrix} \Bbbk & 0 \end{bmatrix}, \begin{bmatrix} 0 & \Bbbk \end{bmatrix}\right) \arrow[dr] & \\
\left(\begin{bmatrix} \Bbbk & \Bbbk \end{bmatrix} \oplus \begin{bmatrix} 0 & \Bbbk \end{bmatrix}, 0\right) \arrow[ur] \arrow[dr] & & & & \left(0, \begin{bmatrix} \Bbbk & \Bbbk \end{bmatrix} \oplus \begin{bmatrix} 0 & \Bbbk \end{bmatrix}\right). \\
 & & \left(\begin{bmatrix} 0 & \Bbbk \end{bmatrix}, \begin{bmatrix} \Bbbk & \Bbbk \end{bmatrix}\right) \arrow[ur] & &
\end{tikzcd}$$

It is proved in [AIR] that this graph is isomorphic to the *Hasse quiver* of *functorially finite torsion classes over A* ordered by inclusion, via the map $(X, P) \mapsto \mathsf{Fac}\,X$. Recall that a *torsion class* $\mathscr{T} \subseteq \operatorname{mod} A$ is a full subcategory closed under extensions and quotients. The full subcategory $\mathsf{Fac}\,X \subseteq \operatorname{mod} A$ consists of $M \in \operatorname{mod} A$ such that there is a surjective morphism $X^n \twoheadrightarrow M$ for some $n \in \mathbb{Z}_{>0}$. A torsion class $\mathscr{T}$ is functorially finite if it is of the form $\mathsf{Fac}\,X$ where X is Ext-projective in $\mathscr{T}$.

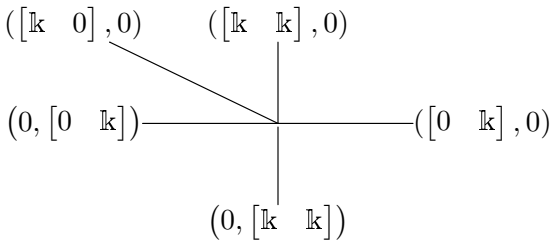
A particular case of τ-tilting pairs consists of pairs $(T, 0)$ where T is *tilting*. Algebras with finitely many tilting modules have been studied by Riedtmann–Schofield [RS], Unger [Ung] and Ingalls–Thomas [IT]. In [DIJ] and [DIR$^+$, DIRT, DIP], we study the case when the number of τ-tilting pairs is finite. We give this characterization:

Theorem A (Theorem 3.1)**.** *The algebra A has finitely many isomorphism classes of basic τ-tilting pairs if and only if all torsion classes over A are functorially finite.*

In particular, thanks to the bijection $(X, P) \mapsto \mathsf{Fac}\, X$ given above, we can reformulate Theorem A in the following way. The algebra A has finitely many functorially finite torsion classes if and only if all torsion classes over A are functorially finite. It implies that A has finitely many torsion classes if and only if all torsion classes over A are functorially finite.

In particular, in this case, the set $\mathsf{tors}\, A$ of torsion classes over A is a finite lattice. If $I \subseteq A$ is an ideal and $B = A/I$, we get that $\pi : \mathsf{tors}\, A \to \mathsf{tors}\, B$, $\mathscr{T} \mapsto \mathscr{T} \cap \mathsf{mod}\, B$ is a surjective morphism of lattices. In Subsections 3.1 to 3.3, we give fundamental tools to understand π. More precisely, we give a "categorification" of lattice quotients of $\mathsf{tors}\, A$ (in the sense that we give an algebraic interpretation of possible quotients) and we study conditions for a lattice quotient of $\mathsf{tors}\, A$ to be of the form $\mathsf{tors}\, B$. We apply these results to give new, categorical, proofs of results about *Cambrian lattices* in Subsection 3.4

In [DIJ], we introduce the combinatorial notion of the $\mathbf{g}$-*vecteur* $\mathbf{g}^X$ associated to a τ-rigid module X. Namely, $\mathbf{g}^X = [P_0] - [P_1]$ in the Grothendieck group $K_0(\mathsf{proj}\, A)$ for a minimal projective presentation $P_1 \to P_0 \to X \to 0$ of X. We define also the $\mathbf{g}$-vector of a pair $(0, P)$ with P projective to be $-[P]$. Then we define the cone of a τ-tilting pair to be the cone generated by the $\mathbf{g}$-vectors of its summands. These cones behave well and form a so-called *fan*, being dual of the exchange graph Γ_A. For instance, for the example given before, this fan is:



where 1-dimensional cones have been labelled. The τ-tilting pairs correspond to the five areas delimited by half-lines. These results generalize [Hil] for tilting modules and [Pla] for cluster-tilting modules. See Subsection 3.5 for more details. See also Section 5 for the link with cluster-tilting theory.

Categorification of cluster algebra using Cohen-Macaulay modules. In 2001, Fomin–Zelevinsky have introduced a new class of algebras called *cluster algebras* [FZ1, FZ2] in order to give an appropriate framework to study canonical bases and total positivity [Lus1, Lus2]. Berenstein–Fomin–Zelevinsky have shown that coordinate rings of numerous algebraic varieties attached to complex semi-simple Lie groups are endowed with the structure of a cluster algebra [BFZ].

Various articles [MRZ, BMR$^+$, CC, GLS1, ...] have shown that cluster algebras can be often *categorified* by triangulated categories or exact categories. My PhD thesis [Dem1, Dem2, Dem3] consisted in adding a group action to some of these categorifications to obtain so-called *skew-symmetrizable* cluster algebras. Let us give a quick overview of additive categorifications of cluster algebras. We consider a triangulated or an exact category $\mathscr{C}$ having the property that $\mathsf{Ext}^1_{\mathscr{C}}(X, Y) \cong \mathsf{D}\,\mathsf{Ext}^1_{\mathscr{C}}(Y, X)$ functorially in X and Y (see Subsection 1.3 for more details about exact categories). Then, we consider a class $\mathscr{O}$ of objects of $\mathscr{C}$, called (basic) *cluster-tilting*, having the property that for any $T \in \mathscr{O}$ and any non-projective indecomposable summand X of T, there exists exactly one indecomposable $Y \in \mathscr{C}$ non-isomorphic to X up to isomorphism such that $\mu_X(T) := T/X \oplus Y \in \mathscr{O}$. We call the operation μ_X a *mutation*. We consider the *exchange graph* Γ with set of vertices the set of isomorphism classes of objects in $\mathscr{O}$ and edges

corresponding to mutations. Comparing with the previous subsection, the exchange graph is not any more oriented. We can recover an orientation of Γ by choosing a source $T \in \mathscr{O}$. Then the orientation is canonical (see Section 5 for more details).

We say that $\mathscr{C}$ *categorifies* a cluster algebra $\mathcal{A}$ if Γ is isomorphic to the *exchange graph* of $\mathcal{A}$ with some compatibility conditions. More precisely, we require the existence of a *cluster character*, that is a map $\varphi : \mathsf{add}\,\mathscr{O} \to \mathscr{A}$ that "maps" mutation in $\mathscr{O}$ to mutation in $\mathcal{A}$ in a strong sense explained in Section 4 (here, $\mathsf{add}\,\mathscr{O}$ consists of all finite direct sums of direct summands of objects in $\mathscr{O}$). Several cluster character have already been introduced, for example in [CC, Pal, FK, GLS2, Dem3, ...]. Note that, in practice, the cluster character often naturally extends to the whole category $\mathscr{C}$, and some properties of φ on $\mathsf{add}\,\mathscr{O}$ can be extended as well.

In Section 4, we explain two methods of categorification of cluster algebras using exact categories of *Cohen-Macaulay modules over orders*. In the first case, Subsection 4.1, the cluster algebras we categorify come from the combinatorics of triangulations of a polygon (or a once-punctured polygon) with *flips* as mutation. In this case, sides of the polygon cannot be flipped and correspond to projective objects in the categorification. Notice that in the case of a non-punctured polygon, we recover the Plücker coordinates of the Grassmannian of 2-dimensional planes, mutations in the cluster algebra being classical Plücker relations. In the second case, Subsection 4.2, we generalize this idea to categorify coordinate rings of partial flag varieties, extending in a certain way works by Geiss–Leclerc–Schröer [GLS2]. Of course, in this case, we lose the combinatorial model of polygons. Meanwhile, we get important exact equivalences of categories of the form $\mathscr{E}' \cong \mathscr{E}/[P]$ where $\mathscr{E}$ and $\mathscr{E}'$ are exact categories and P is a projective-injective object. Notice that this kind of equivalence, to the best of our knowledge, have not been widely studied so far, except in [JKS] which our work generalizes. On a more abstract level, Chen gives in [Che] some criterion for a category $\mathscr{E}/[\mathscr{F}]$ to be exact, which we generalize in [DI1, §3B]. In Subsection 4.1, we do not describe explicitly a cluster character, but it is elementary to extend combinatorially the one of [CC] to this situation. In Subsection 4.2, we describe a cluster character which is a combinatorial extension of the one of [GLS2].

Algebras of partial triangulations. In Section 6, we introduce a new class of algebras, defined from *partial triangulations* of surfaces with marked points. This class contains both important classes of *Jacobian algebras of surfaces* and *Brauer graph algebras*. Jacobian algebras appear naturally in some additive categorifications of cluster algebras as endomorphism rings of cluster-tilting objects. Brauer graph algebras appear in the study of modular representations of groups. These two classes share important properties that we aim to generalize to this new bigger class.

For instance, these algebras always have finite rank over the base ring (this rank is computed explicitly). If the base ring is an algebraically closed field then these algebra are of tame representation type. We also define *flips* of partial triangulations which give *derived equivalences* between algebras related by a flip.

1. Modules and categories

While we suppose that the reader is aware of usual category theory (especially additive, abelian and triangulated categories), we spend a few pages to recall concepts which are central in this document. If it is not stated otherwise, all considered subcategories are full.

1.1. **Module categories.** In this short subsection, we fix notations about module categories. Let A be a (non-necessarily commutative) ring with unit. In this document, the expression *ideal of* A means "two-sided ideal of A". A (right) A-*module* is an (additive) abelian group X with a right action of A such that 1_A acts as the identity of X. Morphisms of A-modules are morphisms of additive groups that commute with the action of A. We denote the category of A-modules by $\mathsf{Mod}\, A$. The category $\mathsf{Mod}\, A$ is an *abelian category*, *i.e.* it possesses well-behaved kernels and cokernels. We say that $X \in \mathsf{Mod}\, A$ is *finitely generated* if there exists a finite subset $\mathscr{X} \subseteq X$ such that X is the smallest submodule of X containing $\mathscr{X}$. We denote the full subcategory of finitely generated A-modules by $\operatorname{mod} A$. If A is Noetherian (*i.e.* if there is no infinite increasing sequence of ideals in A), then $\operatorname{mod} A$ is also abelian. Notice that the path algebras $\Bbbk Q$ defined below are not necessarily Noetherian, as well as the algebras constructed in Section 6. However, all algebras A such that we consider $\operatorname{mod} A$ in this document are Noetherian. This is the case when, for example, $\Bbbk$ is Noetherian and A is finitely generated as a $\Bbbk$-module.

Most of the time in this document, we will suppose that A is a $\Bbbk$-algebra for a unital commutative ring $\Bbbk$. In this case, there is a canonical morphism $\Bbbk \to A$ mapping $1_\Bbbk$ to 1_A, hence there is a forgetful functor from $\mathsf{Mod}\, A$ to $\mathsf{Mod}\, \Bbbk$ and a A-module will be considered implicitly with its structure of $\Bbbk$-module also. In particular, if $\Bbbk$ is a field, a A-module is a $\Bbbk$-vector space and morphisms of A-modules are $\Bbbk$-linear. Notice that, if A is finitely generated as a $\Bbbk$-module, then the forgetful functor $\mathsf{Mod}\, A \to \mathsf{Mod}\, \Bbbk$ restricts to a forgetful functor $\operatorname{mod} A \to \operatorname{mod} \Bbbk$. It is the case when A is a finite-dimensional $\Bbbk$-algebra: $\operatorname{mod} A$ consists of finite-dimensional A-modules.

As a special case of $\Bbbk$-algebras, we recall the definition of path algebras. A *quiver* $Q = (Q_0, Q_1)$ is a finite graph with set of vertices Q_0 and set of arrows Q_1. The *path algebra* $\Bbbk Q$ has a basis consisting of paths of Q (including zero-length paths, in bijection with Q_0), and a bilinear product defined on paths by

$$\omega_1\omega_2 = \begin{cases} \text{the concatenation of } \omega_1 \text{ and } \omega_2 \text{ if } \omega_1 \text{ ends where } \omega_2 \text{ starts;} \\ 0 \text{ else.} \end{cases}$$

It has a unit $1_{\Bbbk Q}$ which is the sum of all zero-length paths. An alternative description of $\mathsf{Mod}\, \Bbbk Q$ is the following one. We denote by $\mathsf{Rep}_\Bbbk\, Q$ the category whose objects X consist of a pairs $((X_i)_{i\in Q_0}, (X_q)_{q\in Q_1})$, where for each i, $X_i \in \mathsf{Mod}\, \Bbbk$ and for each arrow $q : i \to j$, X_q is a morphism of $\Bbbk$-modules from X_i to X_j. Then a morphism f from X to Y is a family $(f_i)_{i\in Q_0}$, where $f_i : X_i \to Y_i$ is a morphism of $\Bbbk$-modules and for each arrow $q : i \to j$, the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[r, "X_q"] \arrow[d, "f_i"'] & X_j \arrow[d, "f_j"] \\ Y_i \arrow[r, "Y_q"'] & Y_j. \end{tikzcd}$$

This category is called the category of $\Bbbk$-representations of Q. It is an elementary fact that $\mathsf{Mod}\, \Bbbk Q \cong \mathsf{Rep}_\Bbbk\, Q$. Moreover, for any ideal $I \subseteq \Bbbk Q$, the full subcategory $\mathsf{Mod}(\Bbbk Q/I)$ of $\mathsf{Mod}\, \Bbbk Q$ is easy to understand as a full subcategory of $\mathsf{Rep}_\Bbbk\, Q$ via this equivalence (we can make sense of a representation of quiver respecting a relation). If $\Bbbk Q/I$ is finite generated as a $\Bbbk$-module, then $\operatorname{mod}(\Bbbk Q/I)$ is equivalent to the full subcategory of $\mathsf{Rep}_\Bbbk\, Q$ consisting of X respecting relations in I and having $X_i \in \operatorname{mod} \Bbbk$ for all i.

We call *admissible ideal* of $\Bbbk Q$ an ideal that is generated by linear combinations of paths of length at least 2 and that contains all path of length n for a sufficiently large integer n. This class of algebra is important thanks to the following key theorem due to Gabriel:

Theorem 1.1 ([Gab1, Gab2, Gab3, ASS]). *If $\Bbbk$ is an algebraically closed field, then for any finite dimensional $\Bbbk$-algebra A, there exist a quiver Q, unique up to isomorphism, and a (non-unique) admissible ideal $I \subseteq \Bbbk Q$, such that there are equivalences of categories*

$$\operatorname{mod} A \cong \operatorname{mod}(\Bbbk Q/I) \quad \textit{and} \quad \mathsf{Mod}\, A \cong \mathsf{Mod}(\Bbbk Q/I).$$

Thus, for representation theoretical problems, we often identify the class of finite dimensional algebras with the class of admissible quotients of path algebras of quivers. In Theorem 1.1, Q is usually called the *Gabriel quiver of* A or simply the *quiver of* A.

Another case of interest for this document consist of Cohen-Macaulay modules. We restrict here to the very special case where A is an R-*order* for $R = \Bbbk[t]$ or $R = \Bbbk[\![t]\!]$, namely A is an R-algebra that is free of finite rank as an R-module. Notice that our definition is not completely in line with the usual one [AG, Sim]. Indeed, it is usually assumed that R is complete, which is not the case for $\Bbbk[t]$ and that $\mathbb{K} \otimes_R A$ is semisimple where $\mathbb{K}$ is the fraction field of R. While the category of A-modules is usually huge, it makes sense to restrict to the category $\mathsf{CM}\, A$ of Cohen-Macaulay A-modules, *i.e.* the full subcategory of $\operatorname{mod} A$ consisting of A-modules that are free of finite rank as R-modules. Notice that $\mathsf{CM}\, A$ is not abelian, but only an exact category as it will be explained in Subsection 1.3. We refer to [AG, Aus1, CR, Sim, Yos] for more details about orders and Cohen-Macaulay modules.

Finally, we introduce a slight generalization of module categories over rings. Let $\mathscr{C}$ be an additive category. Then we call $\mathsf{Mod}\,\mathscr{C}$ the category of covariant functors F from $\mathscr{C}$ to the category $\mathcal{A}b$ of abelian groups. A morphism φ from F to G is then a *natural transformation* from F to G, namely a family of morphisms $\varphi_X : FX \to GX$ in $\mathcal{A}b$ indexed by objects X of $\mathscr{C}$, satisfying the following commutation for any morphism $f : X \to Y$ in $\mathscr{C}$:

$$\begin{tikzcd} FX \arrow[r, "\varphi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\varphi_Y"'] & GY. \end{tikzcd}$$

It is again an abelian category. It is a generalization of $\mathsf{Mod}\, A$. Indeed, consider the category $\mathscr{C}_A$ with class of objects $\mathbb{Z}_{\geq 0}$ and morphism spaces $\mathsf{Hom}_{\mathscr{C}_A}(m,n) := \mathsf{M}_{n,m}(A^{\mathsf{op}})$ with obvious composition ($\mathsf{M}_{n,m}(A^{\mathsf{op}})$ is the space of $n \times m$ matrices with coefficients in A^{op}). Then, $\mathscr{C}_A$ is additive and there is a canonical equivalence of category $\mathsf{Mod}\, A \to \mathsf{Mod}\,\mathscr{C}_A$ mapping an A-module X to the functor $F_X : n \mapsto X^n$ with, for $M \in \mathsf{M}_{n,m}(A)$, $F_X M := [- \cdot M_{i,j}]_{1 \leq i \leq n, 1 \leq j \leq m}$. As an alternative, if Q is a quiver and $\Bbbk$ a commutative ring, we can consider the free additive category $\mathscr{C}_{\Bbbk,Q}$ generated over $\Bbbk$ by indecomposable objects Q_0 and morphisms $j \to i$ for each arrow $i \to j \in Q_1$. Then, $\mathsf{Mod}\,\Bbbk Q$ is equivalent to $\mathsf{Mod}\,\mathscr{C}_{\Bbbk,Q}$. There is also a natural way to pass to the quotient by an ideal I of $\Bbbk Q$.

1.2. **Approximations.** Let $\mathscr{C}$ be an additive category. We say that a monomorphism $f : X \to Y$ (respectively, an epimorphism $g : Y \to X$) in $\mathscr{C}$ *splits* if there exists $g : Y \to X$ (respectively, $f : X \to Y$) such that $gf = \mathsf{Id}_X$. We say that a split monomorphism f (respectively, a split epimorphism g) is *non-trivial* it is not an isomorphism. A monomorphism $f : X \to Y$ (respectively, an epimorphism $g : Y \to X$) is split if and only if there is an isomorphism $\varphi : X \oplus Y' \to Y$ such that $f = \varphi \circ \iota$ where $\iota : X \to X \oplus Y'$ is the canonical inclusion (respectively, $g = \pi \circ \varphi^{-1}$ where $\pi : X \oplus Y' \to X$ is the canonical projection).

For $X \in \mathscr{C}$, we also say that an idempotent $e \in \mathsf{End}_{\mathscr{C}}(Y)$ *splits* if $e = fg$ where $f : X \to Y$ is a split monomorphism and $g : Y \to X$ is a split epimorphism. It is equivalent to say that there exists an isomorphism $\varphi : X \oplus Y' \to Y$ such that $e = \varphi \left[\begin{smallmatrix} \mathsf{Id}_X & 0 \\ 0 & 0 \end{smallmatrix}\right] \varphi^{-1}$. We say that $\mathscr{C}$ is *idempotent complete* if all idempotent endomorphisms in $\mathscr{C}$ split. As it is an elementary observation that any additive category $\mathscr{C}$ embeds canonically in a (minimal) idempotent complete one, we will only consider idempotent complete additive categories.

We recall that $\mathscr{C}$ is a *Krull-Schmidt* category if for any indecomposable $X \in \mathscr{C}$, $\mathsf{End}_{\mathscr{C}}(X)$ is a *local ring* (*i.e.* it has a unique maximal ideal). As a typical example, we can consider the category

$\mathscr{C} = \operatorname{mod} A$ of finite-dimensional modules over a finite-dimensional $\Bbbk$-algebra A. More generally, if $\mathscr{C}$ is linear over a field $\Bbbk$ and all morphism spaces of $\mathscr{C}$ are finite dimensional, then it is Krull-Schmidt (as we supposed already that it is idempotent complete). Being Krull-Schmidt implies the following property, sometimes called also Krull-Schmidt property: if $X_1 \oplus X_2 \oplus \cdots \oplus X_m \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n$ for some indecomposable $X_1, \ldots, X_m, Y_1, \ldots, Y_n \in \mathscr{C}$, then $m = n$ and there is a permutation $\sigma \in \mathfrak{S}_n$ satisfying $Y_i \cong X_{\sigma(i)}$ for any i. Notice that the converse is not true as we see by considering the category $\mathsf{CM}(\Bbbk[t])$ of free modules of finite rank over the polynomial ring $\Bbbk[t]$ (see also Subsection 4.1).

We define minimal morphisms:

Definition 1.2. (a) We say that $f : X \to Y$ in $\mathscr{C}$ is *left minimal* if for any $g \in \mathsf{End}_{\mathscr{C}}(Y)$ such that $fg = f$, g is invertible.
(b) We say that $g : Y \to Z$ in $\mathscr{C}$ is *right minimal* if for any $f \in \mathsf{End}_{\mathscr{C}}(Y)$ such that $fg = g$, f is invertible.

If $\mathscr{C}$ is additionally Krull-Schmidt, then $f : X \to Y$ is left minimal if and only if it does not factor through a non-trivial split monomorphism $i : Y' \to Y$. In the same way $g : Y \to Z$ is right minimal if and only if it does not factor through a non-trivial split epimorphism $\pi : Y \to Y'$.

The following concepts are central in this document:

Definition 1.3. Let $\mathscr{D} \subseteq \mathscr{C}$ an additive subcategory.
(a) We say that $f : X \to Y$ in $\mathscr{C}$ is a *left $\mathscr{D}$-approximation* (of X) if $Y \in \mathscr{D}$ and any morphism from X to any object of $\mathscr{D}$ factors through f, or equivalently $\mathsf{Hom}_{\mathscr{C}}(f, Y') : \mathsf{Hom}_{\mathscr{C}}(Y, Y') \to \mathsf{Hom}_{\mathscr{C}}(X, Y')$ is surjective for any $Y' \in \mathscr{D}$.
(b) We say that $g : Y \to Z$ in $\mathscr{C}$ is a *right $\mathscr{D}$-approximation* (of Z) if $Y \in \mathscr{D}$ and any morphism from any object of $\mathscr{D}$ to Z factors through g, or equivalently $\mathsf{Hom}_{\mathscr{C}}(Y', g) : \mathsf{Hom}_{\mathscr{C}}(Y', Y) \to \mathsf{Hom}_{\mathscr{C}}(Y', Z)$ is surjective for any $Y' \in \mathscr{D}$.

Notice that, in Definition 1.3, if $\mathscr{C}$ is Krull-Schmidt and an object $X \in \mathscr{C}$ admits a left (respectively, right) $\mathscr{D}$-approximation, then it admits a left (respectively right) minimal $\mathscr{D}$-approximation which is unique up to isomorphism.

Finally, we define the following more abstract concepts, which can be seen as relative finiteness conditions concerning certain subcategories of $\mathscr{C}$:

Definition 1.4. Consider an additive subcategory $\mathscr{D} \subseteq \mathscr{C}$.
(a) We say that $\mathscr{D}$ is *covariantly finite* if any $X \in \mathscr{C}$ admits a left $\mathscr{D}$-approximation.
(b) We say that $\mathscr{D}$ is *contravariantly finite* if any $Z \in \mathscr{C}$ admits a left $\mathscr{D}$-approximation.
(c) We say that $\mathscr{D}$ is *functorially finite* if it is covariantly finite and contravariantly finite.

1.3. **Exact categories.** In this subsection, we briefly review the essential properties of exact categories. This concept generalize the notion of short exact sequences found in abelian categories without requiring the existence of kernels and cokernels. Thus, we can continue to do homological algebra in this wider setting. For more details, we refer to [Büh], [Kel, Appendix A] and [Qui]. Let $\mathscr{E}$ be an additive category. Let us call *short exact sequences* of $\mathscr{E}$ pairs of morphisms (i, d) such that i is a kernel of d and d a cokernel of i. Let $\mathscr{S}$ be a class of short exact sequences of $\mathscr{E}$, closed under isomorphisms, direct sums and direct summands. If a short exact sequence (i, d) is in $\mathscr{S}$, we call it a *conflation* and denote it by

$$0 \to X \xrightarrow{i} Y \xrightarrow{d} Z \to 0,$$

and we call i an *inflation* and d a *deflation*. The class $\mathscr{S}$ is called an *exact structure* on $\mathscr{E}$ (and $\mathscr{E}$ is said to be an *exact category*) if it satisfies the following properties, equivalent to the original axioms:

(a) Identity morphisms are inflations and deflations.
(b) The composition of two inflations (resp. deflations) is an inflation (resp. deflation).

(c) If $0 \to X \xrightarrow{i} Y \xrightarrow{d} Z \to 0$ is a conflation, for any morphisms $f : Z' \to Z$ and $g : X \to X'$, there exist commutative diagrams

$$\begin{tikzcd}
\xi f : 0 \arrow[r] & X \arrow[r] \arrow[d, equal] & Y' \arrow[r, "d'"] \arrow[d, "f'"'] \arrow[dr, phantom, "(1)"] & Z' \arrow[r] \arrow[d, "f"] & 0 \\
\xi : 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "d"'] & Z \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
\xi : 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "(2)"] & Y \arrow[r] \arrow[d, "g'"] & Z \arrow[r] \arrow[d, equal] & 0 \\
g\xi : 0 \arrow[r] & X' \arrow[r, "i'"'] & Y' \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

the rows of which are conflations, the square (1) being called a *pull-back* and (2) being called a *push-out*.

Example 1.5. Typical examples of exact categories in this documents are:

(a) Categories $\mathsf{Mod}\, A$ or $\mathrm{mod}\, A$ for a ring A.
(b) Categories $\mathsf{CM}\, A$ where A is an R-order as defined in Section 1.1.
(c) More generally full subcategories $\mathscr{E}$ of abelian categories $\mathscr{C}$ that are closed under extensions (*i.e.* for any short exact sequence $0 \to X \to Y \to Z \to 0$ of $\mathscr{C}$, if $X, Z \in \mathscr{E}$ then $Y \in \mathscr{E}$). In this case, we will always endow $\mathscr{E}$ with the exact structure which consists of short exact sequences of $\mathscr{C}$ with three terms in $\mathscr{E}$.

We deduce easily from the axioms that:

- The class $\mathscr{S}$ is stable by isomorphisms.
- Split monomorphisms are inflations and split epimorphisms are deflations.
- In (c), we have the following conflations:

$$0 \to Y' \xrightarrow{\left[\begin{smallmatrix} -f' \\ d' \end{smallmatrix}\right]} Y \oplus Z' \xrightarrow{[\,d\ f\,]} Z \to 0 \quad \text{and} \quad 0 \to X \xrightarrow{\left[\begin{smallmatrix} -g \\ i \end{smallmatrix}\right]} X' \oplus Y \xrightarrow{[\,i'\ g'\,]} Y' \to 0.$$

- In (c), f (respectively, g) is an inflation if and only if f' (respectively, g') is; in this case, d (respectively, i') identify $\mathsf{coker}\, f$ and $\mathsf{coker}\, f'$ (respectively, $\mathsf{coker}\, g$ and $\mathsf{coker}\, g'$).
- In (c), f (respectively, g) is an deflation if and only if f' (respectively, g') is; in this case, d' (respectively, i) identify $\mathsf{ker}\, f$ and $\mathsf{ker}\, f'$ (respectively, $\mathsf{ker}\, g$ and $\mathsf{ker}\, g'$).
- If a morphism is an inflation and a deflation, then it is an isomorphism.
- If, in a morphism of conflation, the left and right components are inflations (respectively, conflations), then the middle one is.
- In (c), the diagrams are uniquely determined up to unique isomorphisms.

An object $P \in \mathscr{E}$ is said to be *projective* if for any deflation $d : Y \to Z$, $\mathsf{Hom}_{\mathscr{E}}(P, d)$ is surjective (in other terms, any morphism $P \to Z$ factors through d). A *projective cover* of an object $X \in \mathscr{E}$ is a deflation $P \to X$ where P is projective. An object $I \in \mathscr{E}$ is said to be *injective* if for any inflation $i : X \to Y$, $\mathsf{Hom}_{\mathscr{E}}(i, I)$ is surjective (in other terms, any $X \to I$ factors through i). An *injective envelope* of an object $X \in \mathscr{E}$ is an inflation $X \to I$ where I is injective. The subcategory of projective (respectively, injective) objects is denoted by $\mathscr{P}$ (respectively, $\mathscr{I}$). As for abelian categories, $\mathscr{E}$ is said to have *enough injective objects* (resp. *enough projective objects*) if every object admits an injective envelope (respectively, a projective cover).

All these notions permit to define extension bifunctors $\mathsf{Ext}^i_{\mathscr{E}}$ which satisfy the expected properties, either from Yoneda's structure of admissible long exact sequences, or using projective resolutions if $\mathscr{E}$ has enough projective objects, or using injective resolutions if $\mathscr{E}$ has enough injective objects, or more generally using the derived category of $\mathscr{E}$. More precisely, $\mathsf{Ext}^1_{\mathscr{E}}(Z, X)$ parametrizes conflations $0 \to X \to Y \to Z \to 0$, up to equivalence (two conflations are equivalent if there is an isomorphism of the form $(\mathsf{Id}_X, \varphi, \mathsf{Id}_Z)$ between each other). The bifunctor structure of $\mathsf{Ext}^1_{\mathscr{E}}$ can be read in terms of conflations in the two diagrams of axiom (c) above ($\mathsf{Ext}^1_{\mathscr{E}}(f, X)$ maps ξ to ξf and $\mathsf{Ext}^1_{\mathscr{E}}(Z, g)$ maps ξ to $g\xi$).

Throughout, we will use the following definition:

Definition 1.6. Let $\mathscr{E}$ and $\mathscr{E}'$ be exact categories and $F : \mathscr{E} \to \mathscr{E}'$ be a functor. It is *exact* if it maps conflations of $\mathscr{E}$ to conflations of $\mathscr{E}'$. Then it induces morphism of bifunctors $\mathsf{Ext}^1_{\mathscr{E}}(-,-) \to \mathsf{Ext}^1_{\mathscr{E}'}(F-,F-)$. We say that F is *exact bijective* if F is exact and $\mathsf{Ext}^1_{\mathscr{E}}(-,-) \to \mathsf{Ext}^1_{\mathscr{E}'}(F-,F-)$ is an isomorphism. We say that F is an *equivalence of exact categories* if it is an exact bijective equivalence of categories (or, equivalently, an exact equivalence of categories with an exact quasi-inverse).

A typical example of exact bijective functors arises when $\mathscr{E}$ is a full exact subcategory of $\mathscr{E}'$ (*i.e.* a full subcategory which is closed under extensions).

1.4. **Factor categories and stable categories.** Let $\mathscr{C}$ be an additive category and $\mathscr{D}$ be an additive subcategory. We call *factor category* the category $\mathscr{C}/[\mathscr{D}]$ with the objects of $\mathscr{C}$ and $\mathsf{Hom}_{\mathscr{C}/[\mathscr{D}]} := \mathsf{Hom}_{\mathscr{C}}/[\mathscr{D}]$ where $[\mathscr{D}](-,-)$ is the ideal of $\mathsf{Hom}_{\mathscr{C}}(-,-)$ consisting of morphisms factorizing through objects of $\mathscr{D}$. It is also an additive category.

We are interested in more specific cases:

Definition 1.7. If $\mathscr{E}$ is an exact category with enough projective objects (respectively, enough injective objects), we call *stable category of* $\mathscr{E}$ (respectively, *co-stable category of* $\mathscr{E}$) the category $\underline{\mathscr{E}} := \mathscr{E}/[\mathscr{P}]$ (respectively, $\overline{\mathscr{E}} := \mathscr{E}/[\mathscr{I}]$).

To state the following fundamental result, we need the following recollection:

Definition 1.8. If $\mathscr{E}$ is an exact category with enough projective objects, we call *syzygy* of $X \in \mathscr{E}$ the kernel ΩX of a minimal projective cover of X. If $\mathscr{E}$ is an exact category with enough injective objects, we call *co-syzygy* of $X \in \mathscr{E}$ the cokernel $\overline{\Omega} X$ of a minimal injective envelope of X.

It is immediate that Ω is a functor from $\underline{\mathscr{E}}$ to itself and $\overline{\Omega}$ is a functor from $\overline{\mathscr{E}}$ to itself. We recall that an exact category $\mathscr{E}$ is *Frobenius* if it has enough projective objects, enough injective objects and $\mathscr{P} = \mathscr{I}$. In this case, Happel proved

Theorem 1.9 ([Hap]). *If $\mathscr{E}$ is a Frobenius exact category, then $\underline{\mathscr{E}} = \overline{\mathscr{E}}$ has the structure of a triangulated category where triangles are images of conflations and the suspension functor is the co-syzygy functor.*

Remark 1.10. Assume $F : \mathscr{E} \to \mathscr{E}'$ is a dense and exact bijective functor.

(a) For any $X \in \mathscr{E}$, X is projective (respectively, injective) if and only if FX is projective (respectively, injective).
(b) $\mathscr{E}$ has enough projective (respectively, injective) objects if and only if $\mathscr{E}'$ has enough projective (respectively, injective) objects.
(c) $\mathscr{E}$ is Frobenius if and only if $\mathscr{E}'$ is Frobenius.

2. Torsion classes and τ-tilting theory

2.1. **Auslander-Reiten theory.** The aim of this subsection is to introduce fundamental ideas of Auslander-Reiten theory. For more details and proofs of statements, we refer for example to [ARS] or [ASS]. We adopt here a rather axiomatic approach. We consider an exact category $\mathscr{E}$ with Krull-Schmidt property and we define the following fundamental notions:

Definition 2.1. (a) A morphism $f : X \to Y$ in $\mathscr{E}$ is called an *almost split monomorphism* if
- It is a non-split monomorphism;
- X is indecomposable;
- f is left minimal;
- Any $f' : X \to Y'$ that is not a split monomorphism factors through f.

(b) A morphism $g : Y \to Z$ in $\mathscr{E}$ is called an *almost split epimorphism* if
- It is a non-split epimorphism;
- Z is indecomposable;
- g is right minimal;
- Any $g' : Y' \to Z$ that is not a split epimorphism factors through g.

Then, the following fundamental result hold:

Proposition 2.2. (a) *For $X \in \mathscr{E}$ indecomposable, there is at most one almost-split monomorphism $X \to Y$, up to composition with an isomorphism $Y \to Y'$.*

(b) *For $Z \in \mathscr{E}$ indecomposable, there is at most one almost-split epimorphism $Y \to Z$, up to composition with an isomorphism $Y' \to Y$.*

(c) *For a conflation*

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of $\mathscr{E}$, f is an almost-split monomorphism if and only if g is an almost-split epimorphism.

In view of Proposition 2.2, we define:

Definition 2.3. (a) A conflation

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of $\mathscr{E}$ is an *almost-split sequence* or an *Auslander-Reiten sequence* if f is an almost-split monomorphism or equivalently g is an almost-split epimorphism.

(b) We say that $\mathscr{E}$ admits an *Auslander-Reiten theory* if any non-projective indecomposable $Z \in \mathscr{E}$ appears to the right of an Auslander-Reiten sequence and any non-injective indecomposable $X \in \mathscr{E}$ appears to the left of an Auslander-Reiten sequence.

If $\mathscr{E}$ admits an Auslander-Reiten theory, for any Auslander-Reiten sequence $0 \to X \to Y \to Z \to 0$, we define $\tau Z = X$ and $\tau^- X = Z$, which are well-defined up to isomorphism thanks to Proposition 2.2 and we extend them to all objects by additivity. We call τ the *Auslander-Reiten translation*. We also extend these definitions to morphisms using the following commutative diagrams where rows are almost-split sequences:

$$\begin{tikzcd}
0 \arrow[r] & \tau Z \arrow[r] \arrow[d, "\exists \tau v"] & Y \arrow[r] \arrow[d, "\exists"] & Z \arrow[r] \arrow[d, "\forall v"] & 0 \\
0 \arrow[r] & \tau Z' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\forall u"] & Y \arrow[r] \arrow[d, "\exists"] & \tau^- X \arrow[r] \arrow[d, "\exists \tau^- u"] & 0 \\
0 \arrow[r] & \tau X' \arrow[r] & Y' \arrow[r] & \tau^- X' \arrow[r] & 0
\end{tikzcd}$$

While these definitions are not canonical, it is an elementary observation that they become canonical in appropriate stable categories:

Proposition 2.4. *The constructions τ and τ^- induce quasi-inverse equivalences of categories $\tau : \underline{\mathscr{E}} \to \overline{\mathscr{E}}$ and $\tau^- : \overline{\mathscr{E}} \to \underline{\mathscr{E}}$.*

The main sources of categories with Auslander-Reiten theory in this document are given in the following example:

Example 2.5. (a) [ARS] For a finite dimensional $\Bbbk$-algebra A, $\operatorname{mod} A$ admits an Auslander-Reiten theory.

(b) [Aus2, Yos] Consider $R := \Bbbk[\![t]\!]$ and its fraction field $\mathbb{K} := \Bbbk(\!(t)\!)$. For an R-order A, $\mathsf{CM}\, A$ admits an Auslander-Reiten theory if and only if $\mathbb{K} \otimes_R A$ is a semi-simple $\mathbb{K}$-algebra.

Now, we define the *quiver of* $\mathscr{E}$. In order to do so, we have to suppose that $\mathscr{E}$ is $\Bbbk$-linear for a field $\Bbbk$. We also suppose for simplicity here that $\Bbbk$ is algebraically closed. If $X, Y \in \mathscr{E}$ are indecomposable, a morphism $f : X \to Y$ is called *irreducible* if it is not invertible, and if $f = uv$ for two morphisms u and v, then v is a split monomorphism or u is a split epimorphism. Unfortunately, the set of irreducible maps from X to Y do not form a $\Bbbk$-vector space. To avoid this difficulty, we introduce the sub-bifunctor $\mathsf{Rad}_{\mathscr{E}}$ of $\mathsf{Hom}_{\mathscr{E}}$ in the following way: for $X, Y \in \mathscr{E}$, we set

$$\begin{aligned}\mathsf{Rad}_{\mathscr{E}}(X,Y) :&= \{f \in \mathsf{Hom}_{\mathscr{E}}(X,Y) \mid \forall g : Y \to X, \mathsf{Id}_X - gf \text{ is invertible}\}\\ &= \{f \in \mathsf{Hom}_{\mathscr{E}}(X,Y) \mid \forall g : Y \to X, \mathsf{Id}_Y - fg \text{ is invertible}\}.\end{aligned}$$

This functor, called *(Jacobson) radical*, is an *ideal of* $\mathscr{E}$, *i.e.* $\mathsf{Rad}_{\mathscr{E}}(X,Y)$ is a subgroup of $\mathsf{Hom}_{\mathscr{E}}(X,Y)$ and being in the radical is stable by left and right composition by any morphism. If X and Y are indecomposable, $\mathsf{Rad}_{\mathscr{E}}(X,Y)$ is generated by irreducible morphisms from X to Y. Then, we define inductively the powers of the radical: $\mathsf{Rad}^0_{\mathscr{E}} := \mathsf{Hom}_{\mathscr{E}}$ and for any $n \in \mathbb{Z}_{\geq 0}$, $\mathsf{Rad}^{n+1}_{\mathscr{E}}(X,Y) := \sum_{Z \in \mathscr{E}} \mathsf{Rad}^n_{\mathscr{E}}(Z,Y)\, \mathsf{Rad}_{\mathscr{E}}(X,Z) \subseteq \mathsf{Hom}_{\mathscr{E}}(X,Y)$. We finally define the bifunctor $\mathsf{Irr}_{\mathscr{E}}$ by

$$\mathsf{Irr}_{\mathscr{E}}(X,Y) := \mathsf{Rad}_{\mathscr{E}}(X,Y)/\,\mathsf{Rad}^2_{\mathscr{E}}(X,Y)$$

which gives a good approximation of an "irreducible map space". The quiver $Q_{\mathscr{E}}$ of $\mathscr{E}$ has isomorphism classes of indecomposable objects of $\mathscr{E}$ as vertices, and for any two vertices X, Y of $Q_{\mathscr{E}}$, the set of arrows from X to Y form a $\Bbbk$-basis of $\mathsf{Irr}_{\mathscr{E}}(X,Y)$.

If $\mathscr{E}$ admits an Auslander-Reiten theory, then τ induces a bijection, still denoted by τ, from $Q_{\mathscr{E},0} \setminus \mathscr{P}$ to $Q_{\mathscr{E},0} \setminus \mathscr{I}$. In this case, $(Q_{\mathscr{E}}, \tau)$ is called the *Auslander-Reiten quiver of* $\mathscr{E}$. We give two example:

Example 2.6. (a) We consider the quiver $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. Then the Auslander-Reiten quiver of $\operatorname{mod} \mathbb{C}Q = \mathsf{rep}_{\mathbb{C}}\, Q$ is:

$$\begin{tikzcd}
 & & (\mathbb{C} \to \mathbb{C} \to \mathbb{C}) \arrow[dr] & & \\
 & (0 \to \mathbb{C} \to \mathbb{C}) \arrow[ur] \arrow[dr] & & (\mathbb{C} \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[dr] & \\
(0 \to 0 \to \mathbb{C}) \arrow[ur] & & (0 \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[ur] & & (\mathbb{C} \to 0 \to 0) \arrow[ll, dashed]
\end{tikzcd}$$

where we take the convention that arrows from $\mathbb{C}$ to $\mathbb{C}$ are always the identity. The Auslander-Reiten translation τ has been depicted using dashed arrows.

(b) We consider now the algebra $A := \mathbb{C}Q/(\alpha\beta)$. Then the Auslander-Reiten quiver of $\operatorname{mod} A \subset \mathsf{rep}_{\mathbb{C}}\, Q$ is:

$$\begin{tikzcd}
 & (0 \to \mathbb{C} \to \mathbb{C}) \arrow[dr] & & (\mathbb{C} \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[dr] & \\
(0 \to 0 \to \mathbb{C}) \arrow[ur] & & (0 \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[ur] & & (\mathbb{C} \to 0 \to 0) \arrow[ll, dashed]
\end{tikzcd}$$

(c) We take $R := \mathbb{C}[\![t]\!]$ and consider the following R-order, consisting of a subalgebra of $\mathsf{M}_{2,2}(R)$:

$$\Lambda = \begin{bmatrix} R & (t) \\ (t) & R \end{bmatrix}.$$

The Auslander-Reiten quiver of $\mathsf{CM}(\Lambda)$ is:

$$\begin{tikzcd}
 & {[R \quad (t)]} \arrow[dr] & \\
{[(t)} \arrow[ur] \arrow[dr] & {(t)]} \arrow[l, dashed] & {[R} \arrow[u] \arrow[d] & R] \\
 & {[(t)} \arrow[ur] & R]. &
\end{tikzcd}$$

Plain arrows consist of natural inclusions. The two dotted borders are identified.

If $\mathscr{E}$ is a Frobenius category (see Subsection 1.3), then the restriction of $Q_{\mathscr{E}}$ to non-projective objects, endowed with τ, is a so-called *translation quiver*. More generally, the usefulness of the Auslander-Reiten quiver is partially justified by the following remark:

Remark 2.7. For an Auslander-Reiten sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ of $\mathscr{E}$, the number of arrows starting from X (respectively, pointing toward Z) in $Q_{\mathscr{E}}$ is exactly the number of indecomposable summands of Y. More precisely, for a decomposition $Y = \bigoplus_{i=1}^{n} Y_i$ into indecomposable summands, then each $u_i : X \to Y_i$ (respectively, $v_i : Y_i \to Z$) is irreducible and we can choose the arrows of the Auslander-Reiten quiver in such a way that arrows starting from X (respectively, pointing toward Z) are the classes of the u_i (respectively, the v_i).

In particular, $(Q_{\mathscr{E}}, \tau)$ can be seen as a convenient combinatorial tool to encode Auslander-Reiten sequences. Moreover, when Z is indecomposable non-projective, it gives a bijection between arrows of $Q_{\mathscr{E}}$ from Y to Z and arrows from τZ to Y. Finally, notice that almost all arrows of $Q_{\mathscr{E}}$ are in Auslander-Reiten sequences, except arrows going from an injective object to a projective object.

We now give a slightly more conceptual point of view about Auslander-Reiten theory, which is elementary to prove (see the end of Subsection 1.1 for the definition of $\mathscr{E}$-modules):

Proposition 2.8. *The following are equivalent:*

(i) *The category $\mathscr{E}$ admits an Auslander-Reiten theory.*

(ii) *For any $Z \in \mathscr{E} \setminus \mathscr{P}$ indecomposable, the $\mathscr{E}$-module $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$ admits a simple socle which is contained in any non-trivial submodule of $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$. In other terms it admits an injective envelope that is indecomposable.*

(iii) *For any $X \in \mathscr{E} \setminus \mathscr{I}$ indecomposable, the $\mathscr{E}^{\mathsf{op}}$-module $\mathsf{Ext}^1_{\mathscr{E}}(-, X)$ admits a simple socle which is contained in any non-trivial submodule of $\mathsf{Ext}^1_{\mathscr{E}}(-, X)$. In other terms it admits an injective envelope that is indecomposable.*

Moreover, if these hold, then, in (ii), $\mathsf{soc}\,\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$ is supported at the object τZ and is generated by the Auslander-Reiten sequence. In other terms, the Auslander-Reiten sequence is a co-generator *of $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$. In the same way $\mathsf{soc}\,\mathsf{Ext}^1_{\mathscr{E}}(-, X)$ is supported at $\tau^- X$ and is generated by the Auslander-Reiten sequence, which is a co-generator of $\mathsf{Ext}^1_{\mathscr{E}}(-, X)$.*

Remark 2.9. In contrast with Proposition 2.8, if $\mathscr{E}$ has enough projective objects then $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$ is generated as an $\mathscr{E}$-module by the conflation $0 \to \Omega Z \to P_Z \to Z \to 0$ where P_Z is a projective cover of Z. So $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$ admits a projective cover that is usually not indecomposable. In the same way, if $\mathscr{E}$ has enough injective objects then $\mathsf{Ext}^1_{\mathscr{E}}(-, X)$ is generated as an $\mathscr{E}^{\mathsf{op}}$-module by the conflation $0 \to X \to I_X \to \overline{\Omega} X \to 0$ where I_X is an injective envelope of X. So $\mathsf{Ext}^1_{\mathscr{E}}(-, X)$ admits a projective cover that is usually not indecomposable.

One of the important fact from Auslander-Reiten theory is the *Auslander-Reiten duality*:

Theorem 2.10 (*e.g.* [ASS])**.** *We suppose that $\mathscr{E}$ is $\Bbbk$-linear over a field $\Bbbk$ and has an Auslander-Reiten theory. Then for any $X, Z \in \mathscr{E}$,*

(a) *If $\mathscr{E}$ has enough injective objects, $\mathsf{Ext}^1_{\mathscr{E}}(Z, X) \cong \mathsf{D}\,\overline{\mathsf{Hom}}_{\mathscr{E}}(X, \tau Z)$.*

(b) *If $\mathscr{E}$ has enough projective objects, $\mathsf{Ext}^1_{\mathscr{E}}(Z, X) \cong \mathsf{D}\,\underline{\mathsf{Hom}}_{\mathscr{E}}(\tau^- X, Z)$.*

We give a justification of Theorem 2.10. Let $X, Z \in \mathscr{E}$ be indecomposable. We fix a $\Bbbk$-linear map $s_Z : \mathsf{Ext}^1_{\mathscr{E}}(Z, \tau Z) \to \Bbbk$ that does map the Auslander-Reiten sequence ξ_0 to 1. Consider $\xi \in \mathsf{Ext}^1_{\mathscr{E}}(Z, X)$ and $f \in \mathsf{Hom}_{\mathscr{E}}(X, \tau Z)$. Then we have a push-out diagram

$$\begin{tikzcd}
\xi : & 0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
f\xi : & 0 \arrow[r] & \tau Z \arrow[r] & Y' \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

and we put $\langle f, \xi \rangle = s_Z(f\xi)$, which is a bilinear form. As this approach is not the standard one, we give a short prove of the following conclusive lemma:

Lemma 2.11. *The bilinear form $\langle -, - \rangle$ induces a non-degenerate form from $\overline{\mathsf{Hom}}_{\mathscr{E}}(X, \tau Z) \times \mathsf{Ext}^1_{\mathscr{E}}(Z, \tau Z)$ to $\Bbbk$.*

Proof. First of all, if f factors through an injective, if is immediate that $f\xi = 0$, hence $\langle f, \xi \rangle = 0$ for any $\xi \in \mathsf{Ext}^1_{\mathscr{E}}(Z, X)$. If $\xi \in \mathsf{Ext}^1_{\mathscr{E}}(Z, X) \setminus \{0\}$, as ξ_0 is a co-generator of $\mathsf{Ext}^1_{\mathscr{E}}(Z, -)$, there exists $f : X \to \tau Z$ such that $f\xi = \xi_0$ and $\langle f, \xi \rangle = 1$. On the other hand, consider $f : X \to \tau Z$ that does not factor through an injective object. Take $\xi_1 : 0 \to X \to I_X \to \overline{\Omega} X \to 0$ where I_X is an injective envelope of X. As f does not factor through I_X, $f\xi_1 \neq 0$. As ξ_0 is a co-generator of $\mathsf{Ext}^1_{\mathscr{E}}(-, \tau Z)$ and $f\xi_1 \in \mathsf{Ext}^1_{\mathscr{E}}(\overline{\Omega} X, \tau Z) \setminus \{0\}$, there exists $u : Z \to \overline{\Omega} X$ such that $f\xi_1 u = \xi_0$. Then $\langle f, \xi_1 u \rangle = 1$. $\square$

We actually constructed a non-degenerate bilinear functorial form from $\overline{\mathsf{Hom}}_{\mathscr{E}}(-, \tau Z) \times \mathsf{Ext}^1_{\mathscr{E}}(Z, -)$ to $\Bbbk$, hence the isomorphism is functorial in X. In the same way we could construct it functorial in Z. It is proven, at least in good cases, that these isomorphisms can be made bifunctorial (*e.g.* in cases of Example 2.5).

2.2. **Torsion pairs.** Here, we recall the notion of a *torsion pair* in a module category over a finite dimensional $\Bbbk$-algebra A over a field $\Bbbk$. It is a fundamental tool to study *tilting theory* and more generally derived equivalences. We start by giving a rough explanation about tilting theory and derived equivalences. Recall that the bounded derived category $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A)$ is a triangulated category containing in a canonical way $\operatorname{mod} A$ (it is called the *heart* of the canonical t-structure). A celebrated case of derived equivalence $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$ for another finite dimensional algebra B, is called *tilting*. This is the case when there is a *tilting module* $T \in \operatorname{mod} A$ such that $\mathsf{End}_A(T)^{\mathsf{op}} \cong B$ (we take the opposite algebra here as we consider right modules and endomorphisms act left). In such a case, there is a derived equivalence $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$, satisfying the following properties. The pair $(\mathsf{Fac}\, T, T^{\perp})$ is a *torsion pair* in $\operatorname{mod} A$ where

$$\mathsf{Fac}\, T := \{X \in \operatorname{mod} A \mid \exists T^n \twoheadrightarrow X\} \quad \text{and} \quad T^{\perp} := \{X \in \operatorname{mod} A \mid \mathsf{Hom}_A(T, X) = 0\}.$$

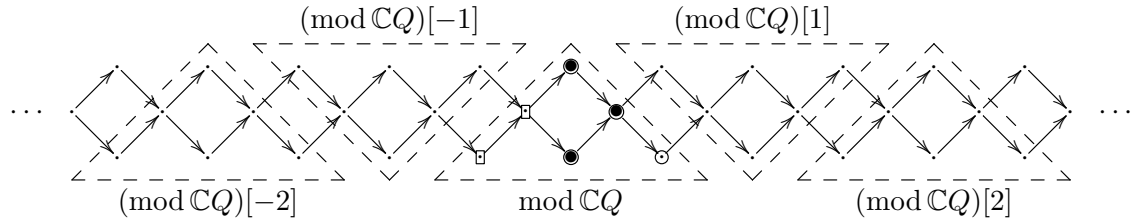
Also, there is a torsion pair $({}^{\perp}(\mathsf{D}\, T), \mathsf{Sub}\, \mathsf{D}\, T)$ in $\operatorname{mod} B$ where $\mathsf{D}\, T := \mathsf{Hom}_{\Bbbk}(T, \Bbbk)$,

$${}^{\perp}(\mathsf{D}\, T) := \{X \in \operatorname{mod} B \mid \mathsf{Hom}_B(X, \mathsf{D}\, T) = 0\} \text{ and } \mathsf{Sub}\, \mathsf{D}\, T := \{X \in \operatorname{mod} B \mid \exists X \hookrightarrow (\mathsf{D}\, T)^n\}.$$

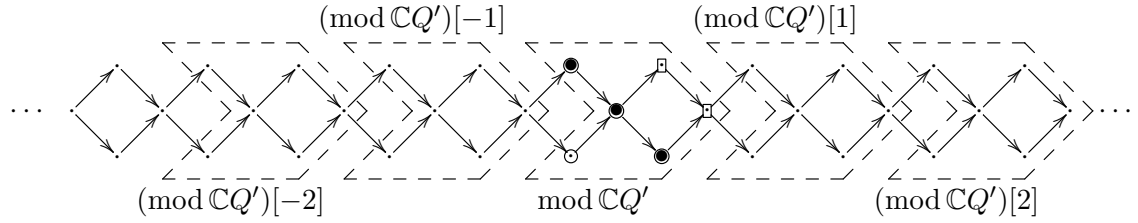
Moreover, inside $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A)$, identified with $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$, we have $\mathsf{Fac}\, T = \mathsf{Sub}\, \mathsf{D}\, T$ and $T^{\perp} = ({}^{\perp}\, \mathsf{D}\, T)[1]$ where $[1]$ is the suspension functor. We illustrate this situation by the following example

Example 2.12. We consider the quiver $Q = 1 \to 2 \to 3$ as in Example 2.6. It turns out that, in this very simple case, called *hereditary*, the indecomposable objects of $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} \mathbb{C}Q)$ are exactly shifted of representations of $\mathbb{C}Q$ and $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} \mathbb{C}Q)$ admits an Auslander-Reiten quiver, we omit

the definition of, which can be depicted as follows:



Moreover, the representation $T := (\mathbb{C} \to \mathbb{C} \to \mathbb{C}) \oplus (\mathbb{C} \to \mathbb{C} \to 0) \oplus (0 \to \mathbb{C} \to 0)$ depicted by thick dots in the picture is tilting and satisfies $\mathsf{End}_{\mathbb{C}Q}(T)^{\mathsf{op}} = \mathbb{C}Q'$ where $Q' = 1' \leftarrow 2' \to 3'$. Indecomposable modules of $\mathsf{Fac}\,T$ have been circled and indecomposable modules of $T^{\perp}$ has been framed. Then $\mathscr{D}^{\mathrm{b}}(\mathbb{C}Q)$ is equivalent with $\mathscr{D}^{\mathrm{b}}(\mathbb{C}Q')$ depicted here:



where $\mathsf{D}\,T = (\mathbb{C} \leftarrow 0 \to 0) \oplus (\mathbb{C} \leftarrow \mathbb{C} \to \mathbb{C}) \oplus (\mathbb{C} \leftarrow \mathbb{C} \to 0)$ has been thicken, $\mathsf{Sub}\,\mathsf{D}\,T = \mathsf{Fac}\,T$ has been circled and ${}^{\perp}(\mathsf{D}\,T) = (T^{\perp})[1]$ has been framed.

Tilting theory was first introduced by Brenner and Butler in [BB2] and generalized in multiple fashions afterwards. See Subsection 2.3 for a generalization. As good classes of derived equivalences are produced by sequences of tilting, studying torsion pairs is a natural step toward understanding or classifying derived equivalences.

More precisely, a full subcategory $\mathscr{T}$ of $\operatorname{mod} A$ is a *torsion class* if it is closed under factor modules and extensions in $\operatorname{mod} A$. Dually, $\mathscr{F} \subseteq \operatorname{mod} A$ is a *torsion-free class* if it is closed under submodules and extensions in $\operatorname{mod} A$. We say that $(\mathscr{T}, \mathscr{F})$ is a *torsion pair* if $\mathscr{T}$ is a torsion class, $\mathscr{F}$ is a torsion-free class, and $\mathscr{F} = \mathscr{T}^{\perp}$ (or, equivalently, $\mathscr{T} = {}^{\perp}\mathscr{F}$). We denote the set of torsion (respectively, torsion-free) classes in $\operatorname{mod} A$ by $\mathsf{tors}\,A$ (respectively, $\mathsf{torsf}\,A$). An alternative characterization of torsion pairs is the following: $\mathscr{T}$ and $\mathscr{F}$ are full-subcategories of $\operatorname{mod} A$ such that for all $T \in \mathscr{T}$ and $F \in \mathscr{F}$, $\mathsf{Hom}_A(T, F) = 0$ and $\mathscr{T} * \mathscr{F} = \operatorname{mod} A$ where $\mathscr{T} * \mathscr{F}$ is the full subcategory of $\operatorname{mod} A$ consisting of X appearing as the middle term of a short exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

We order $\mathsf{tors}\,A$ and $\mathsf{torsf}\,A$ by inclusion. Then there is an anti-isomorphisms of partially ordered set between them, namely $\mathscr{T} \mapsto \mathscr{T}^{\perp}$ and ${}^{\perp}\mathscr{F} \leftarrow\!\shortmid \mathscr{F}$. It turns out that $\mathsf{tors}\,A$ and $\mathsf{torsf}\,A$ are in fact complete lattices. In other terms, any family $(\mathscr{T}_i)_{i \in I}$ of torsion classes admits a *join* $\bigvee_{i \in I} \mathscr{T}_i$, that is a minimum torsion class among classes that are bigger than all $\mathscr{T}_i$'s and a *meet* $\bigwedge_{i \in I} \mathscr{T}_i$, that is a maximum torsion class among classes that are smaller than all $\mathscr{T}_i$'s and the same holds for torsion-free classes. In both cases, the meet is easy to construct: $\bigwedge_{i \in I} \mathscr{T}_i = \bigcap_{i \in I} \mathscr{T}_i$, and the join is obtained by using the anti-isomorphism above: $\bigvee_{i \in I} \mathscr{T}_i = {}^{\perp}\left(\bigcap_{i \in I} \mathscr{T}_i^{\perp}\right)$. Recall that the Hasse quiver $\mathsf{Hasse}\,L$ of an partially ordered set L has set of vertices L and an arrow $x \to y$ if and only if $x > y$ and there is no $z \in L$ with $x > z > y$.

Example 2.13. We continue with the quiver $1 \to 2 \to 3$ and the algebra $A := \mathbb{C}Q$. We give the Hasse quiver of $\mathsf{tors}\,A$ in Figure 2.14, using positions of modules in the Auslander-Reiten quiver depicted in Example 2.6.

2.3. **Torsion classes coming from τ-tilting modules.** A convenient way to study torsion classes consists of indexing them by certain modules. Unfortunately, torsion classes are not all of the form $\mathsf{Fac}\,T$ where T is a tilting module. We introduced in Definition 1.4 the notion of a functorially finite subcategory. We denote the set of functorially finite torsion (respectively, torsion-free) classes in $\operatorname{mod} A$ by $\mathsf{f\text{-}tors}\,A$ (respectively, $\mathsf{f\text{-}torsf}\,A$). We say that an A-module

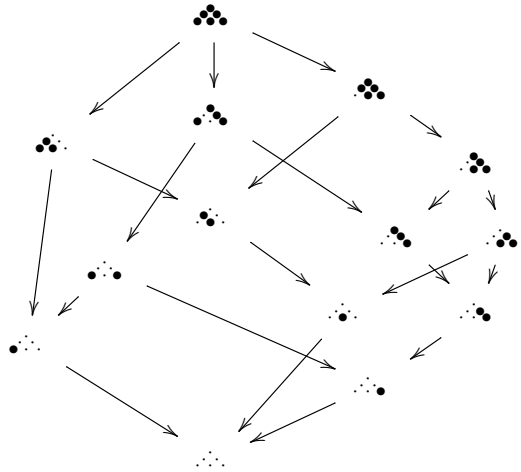


FIGURE 2.14. The lattice $\mathsf{tors}(\mathbb{C}Q)$

$M \in \mathscr{T}$ is Ext-*projective in* $\mathscr{T}$ if for all $N \in \mathscr{T}$ we have $\mathsf{Ext}^1_A(M,N) = 0$. The following result permits to start the investigation:

Proposition 2.15. [AS, Hos, Sma] *Let A be a finite dimensional algebra and $(\mathscr{T},\mathscr{F})$ a torsion pair in* $\operatorname{mod} A$. *The following statements are equivalent:*

(a) *The torsion class $\mathscr{T}$ is functorially finite.*
(b) *The torsion-free class $\mathscr{F}$ is functorially finite.*
(c) *There exist a basic A-module $P(\mathscr{T}) \in \mathscr{T}$ such that $\mathsf{Fac}\, P(\mathscr{T}) = \mathscr{T}$ and $\mathsf{add}\, P(\mathscr{T})$ coincides with the class of Ext-projective A-modules in $\mathscr{T}$.*

If any of the above equivalent conditions hold, then the A-module $P(\mathscr{T})$ is a tilting $(A/\operatorname{ann}\mathscr{T})$-module.

Adachi–Iyama–Reiten [AIR] have started the study of modules $P(\mathscr{T})$ appearing in Proposition 2.15. They are called *support τ-tilting*. We give a brief introduction to τ-tilting theory.

An A-module M is *τ-rigid* if $\mathsf{Hom}_A(M,\tau M) = 0$ where τ is the Auslander-Reiten translation defined in Subsection 2.1. We say that M is *τ-tilting* if, additionally, $|M| = |A|$ holds, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M. Finally, we say that M is *support τ-tilting* if there exists an idempotent e of A such that M is a τ-tilting $(A/(e))$-module. We denote by $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ the set of isomorphism classes of basic support τ-tilting A-modules, by $\tau\text{-}\mathsf{rigid}\, A$ the set of isomorphism classes of τ-rigid A-modules, and by $\mathsf{i}\tau\text{-}\mathsf{rigid}\, A$ the set of isomorphism classes of indecomposable τ-rigid A-modules. By [AIR, §2.7], $M \mapsto \mathsf{Fac}\, M$ is a surjection from $\tau\text{-}\mathsf{rigid}\, A$ to $\mathsf{f}\text{-}\mathsf{tors}\, A$, which restricts to a bijection

$$\mathsf{Fac} : \mathsf{s}\tau\text{-}\mathsf{tilt}\, A \xrightarrow{\sim} \mathsf{f}\text{-}\mathsf{tors}\, A. \tag{2.16}$$

Moreover, this bijection is the inverse of $\mathscr{T} \mapsto P(\mathscr{T})$ in Proposition 2.15.

We also introduce the notion of a *τ-rigid pair*. A τ-rigid pair over A is a pair (M,P) where M is a τ-rigid A-module and P is a projective A-module satisfying $\mathsf{Hom}_A(P,M) = 0$. We say that (M,P) is basic if both M and P are. We denote by $\tau\text{-}\mathsf{rigid\text{-}pair}\, A$ the set of isomorphism classes of τ-rigid pairs over A and by $\mathsf{i}\tau\text{-}\mathsf{rigid\text{-}pair}\, A$ the subset of $\tau\text{-}\mathsf{rigid\text{-}pair}\, A$ consisting of indecomposable ones (*i.e.* $(M,0)$ with M indecomposable or $(0,P)$ with P indecomposable). We identify $M \in \tau\text{-}\mathsf{rigid}\, A$ with $(M,0) \in \tau\text{-}\mathsf{rigid\text{-}pair}$. We say that a τ-rigid pair (M,P) is *τ-tilting* if, in addition, we have $|M| + |P| = |A|$. Notice that it is a maximality condition for the property to be τ-rigid. We denote by $\tau\text{-}\mathsf{tilt\text{-}pair}\, A$ the set of isomorphism classes of basic τ-tilting pair. We have a bijection $\tau\text{-}\mathsf{tilt\text{-}pair}\, A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ mapping (M,P) to M.

We lift the partial order on $\mathsf{f}\text{-}\mathsf{tors}\, A$ to a partial order on $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A \cong \tau\text{-}\mathsf{tilt\text{-}pair}\, A$ via Fac.

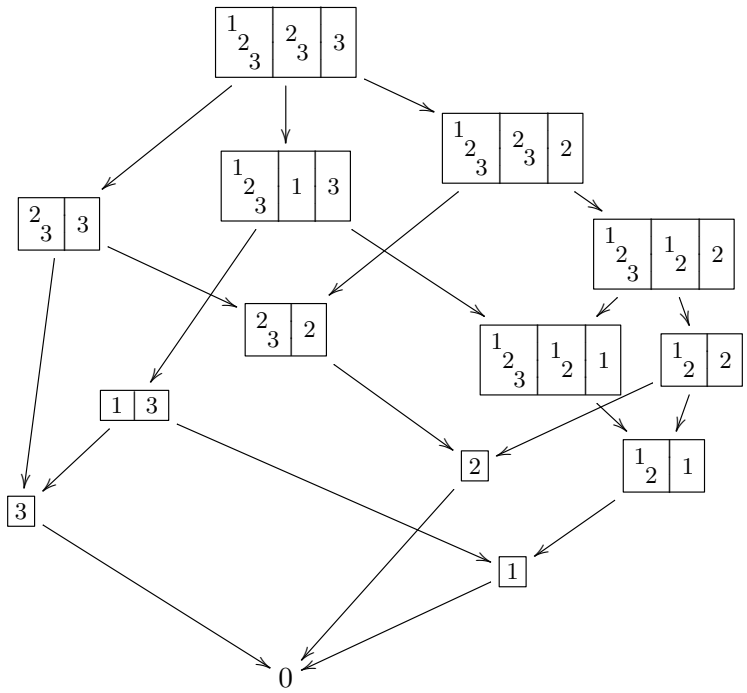


FIGURE 2.17. The lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}(\mathbb{C}Q)$

Example 2.18. We continue with our running example $Q = 1 \to 2 \to 3$. The Hasse quiver $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,\mathbb{C}Q)$ is drawn in Figure 2.17. It can be compared to Figure 2.14. We use here the composition series notation to describe indecomposable modules. Each digits represent a basis vector supported at the corresponding vertex of the quiver and non-zero matrix coefficients of the representation are going from top to bottom.

The order on $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ is characterized in the following way [AIR, Lemma 2.25]: For $(T,P),(U,Q) \in \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$, we have the inequality $(T,P) \geq (U,Q)$ if and only if $\mathsf{Hom}_A(U,\tau T) = 0$ and $\mathsf{Hom}_A(P,U) = 0$.

Moreover, $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A \cong \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ is endowed with a mutation. We call a basic pair $(T,P) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$ *almost τ-tilting* if there exists $(X,Q) \in \mathsf{i}\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$ such that $(T \oplus X, P \oplus Q)$ is τ-tilting.

Theorem 2.19. (a) [AIR, Theorem 2.18] *If (T,P) is an almost τ-tilting pair, there exist exactly two τ-tilting pairs (T_1,P_1) and (T_2,P_2) having (T,P) as a direct summand.*
(b) [AIR, Theorem 2.33] *The Hasse quiver of $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ has an arrow linking (T_1,P_1) and (T_2,P_2) of (a) and all arrows occur in this way.*

An easy consequence of Theorem 2.19 is that $\mathsf{s}\tau\text{-}\mathsf{tilt}\,A$ (equivalently $\mathsf{f}\text{-}\mathsf{tors}\,A$) is Hasse-regular, *i.e.* all vertices of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,A)$ have the same valency. In Theorem 2.19(b), the arrow goes from T_1 to T_2 if and only if $T_1 \notin \mathsf{Fac}\,T_2$. In this case, (T_1,P_1) is the biggest τ-tilting pair having (T,P) as a summand and (T_2,P_2) is the smallest one. We necessarily have $(T_1,P_1) = (T,P) \oplus (X,0)$ for $X \in \mathsf{i}\tau\text{-}\mathsf{rigid}\,A$ and $(T_2,P_2) = (T,P) \oplus (X^*,Q)$ for $(X^*,Q) \in \mathsf{i}\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$, and X^* is uniquely determined by the existence of an exact sequence $X \xrightarrow{u} \overline{T} \to X^* \to 0$, where u is a left minimal $\mathsf{add}(T)$-approximation of X (see Subsection 1.2 for details about approximations).

Theorem 2.19 can be generalized to arbitrary $(X,Q) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$ in the following way:

Theorem 2.20 ([AIR, Theorem 2.10])**.** *If $(X,Q) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\,A$ is basic, there is:*

- *A maximal $(T,P) \in \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\,A$ having (X,Q) as a direct summand. It corresponds to the torsion class ${}^{\perp}(\tau X) \cap Q^{\perp}$ through the bijection* Fac. *We call it the* Bongartz completion of (X,Q).

- *A minimal* $(T, P) \in \tau\text{-}\mathsf{tilt\text{-}pair}\, A$ *having* (X, Q) *as a summand. It corresponds to the torsion class* $\mathsf{Fac}\, X$ *through the bijection* Fac. *We call it the* co-Bongartz completion of X.

3. Lattices of torsion classes and their quotients

If A is a finite dimensional $\Bbbk$-algebra and $B = A/I$ is a quotient of A by an ideal I, then there is a natural embedding $\operatorname{mod} B \subseteq \operatorname{mod} A$ which induces a surjection

$$\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B, \mathscr{T} \mapsto \mathscr{T} \cap \operatorname{mod} B.$$

It turns out that this surjection is in fact a *lattice quotient* (*i.e.* it commutes with meet and join) [DIR$^+$]. The main aim of this section is to understand or even to characterize, in good cases, which lattice quotients of $\mathsf{tors}\, A$ can be realized as $\mathsf{tors}\, B$. Most results are proven in [DIR$^+$, DIRT, DIP] with Iyama, Reading, Reiten and Thomas. A few results come from an earlier paper with Iyama and Jasso [DIJ].

The strategy taken to attack this problem consists in using τ-tilting modules. Unfortunately, as seen in Section 2.3, τ-tilting modules index functorially finite torsion classes and $\mathsf{f\text{-}tors}\, A$ is mostly never a lattice if it is not finite (see [IRTT] for more details). On the other hand, it is always a lattice when it is finite, as justified by the following result:

Theorem 3.1 ([DIJ, Theorem 1.2])**.** *The set* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *is finite if and only if every torsion class of* $\operatorname{mod} A$ *is functorially finite.*

Then, by Theorem 3.1, $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is finite if and only if $\mathsf{f\text{-}tors}\, A = \mathsf{tors}\, A$ if and only if $\mathsf{tors}\, A$ is a finite lattice. We call an algebra A satisfying these conditions *τ-tilting finite*. We focus here on the case where $\mathsf{tors}\, A$ is finite, so that its lattice structure is intimately related with the mutation of τ-tilting pairs. The key point of Theorem 3.1 is the following one: We prove that for $T \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathscr{T} \in \mathsf{tors}\, A$ such that $\mathsf{Fac}\, T \supsetneq \mathscr{T}$ (respectively $\mathscr{T} \supsetneq \mathsf{Fac}\, T$), there is an arrow $T \to U$ (respectively $U \to T$) in $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$ such that $\mathsf{Fac}\, T \supsetneq \mathsf{Fac}\, U \supseteq \mathscr{T}$ (respectively $\mathscr{T} \supseteq \mathsf{Fac}\, U \supsetneq \mathsf{Fac}\, T$) and Theorem 3.1 follows by a combinatorial argument.

In order to study quotients $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B$, which we call *algebraic quotients*, we need first to understand general lattice quotients $\mathsf{tors}\, A \twoheadrightarrow L$. This is the object of the next subsection.

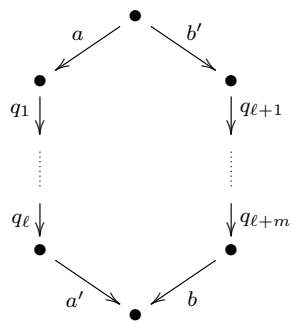
3.1. Lattice quotients of $\mathsf{tors}\, A$.

Let L be a finite lattice. We say that a lattice quotient $\pi : L \twoheadrightarrow L'$ *contracts* an arrow $q : x \to y$ of $\mathsf{Hasse}\, L$ if $\pi(x) = \pi(y)$. It is immediate that π is entirely determined by the set of arrows it contracts. However, the set of arrows contracted by a lattice quotient is subject to constrains. More precisely, we introduce the *forcing relation*. We say that an arrow q *forces* another arrow q' in $\mathsf{Hasse}\, L$, and we denote $q \rightsquigarrow q'$ if for any lattice quotient $\pi : L \twoheadrightarrow L'$, if π contracts q then π also contracts q'. This is clearly a preorder but not a partial order as it possesses non-trivial equivalence classes. We say that q and q' are *forcing-equivalent* if $q \rightsquigarrow q'$ and $q' \rightsquigarrow q$, and we denote $q \leftrightsquigarrow q'$.

Suppose now that A is τ-tilting finite. It turns out that $\mathsf{tors}\, A$ has a very nice structure called *polygonality*:

Proposition 3.2 ([DIR$^+$])**.** *The lattice* $\mathsf{tors}\, A$ *is* polygonal. *In other terms, for any two arrows* $\mathscr{T} \to \mathscr{U}$ *and* $\mathscr{T} \to \mathscr{V}$ *(respectively,* $\mathscr{U} \to \mathscr{T}$ *and* $\mathscr{V} \to \mathscr{T}$*) of* $\mathsf{Hasse}(\mathsf{tors}\, A)$*, the Hasse quiver of the segment* $[\mathscr{U} \wedge \mathscr{V}, \mathscr{T}] := \{\mathscr{S} \mid \mathscr{U} \wedge \mathscr{V} \leq \mathscr{S} \leq \mathscr{T}\}$ *(respectively,* $[\mathscr{T}, \mathscr{U} \vee \mathscr{V}]$*) is a polygon,* i.e. *it consists of its top element, its bottom element, and a disjoint union of exactly two chains.*

As a consequence, we have the following description of the forcing relation on $\mathsf{tors}\, A$: $\rightsquigarrow$ is the transitive closure of the relation $a \rightsquigarrow' b \rightsquigarrow' a$ and $a \rightsquigarrow' q_i$ for all polygons as follows:



3.2. **Brick labelling and categorification of the forcing relation.** One of the main tools to study lattice quotients of $\mathsf{tors}\, A$ is the following *categorification* of the forcing relation defined in Subsection 3.1. We first define the notion of the *brick labelling* of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$. A *brick* is an A-module the endomorphism algebra of which is a division algebra. A set $\{S_i\}_{i\in I}$ of bricks (or its direct sum) is called *semibrick* if $\mathsf{Hom}_A(S_i, S_j) = 0$ for any $i \neq j$. We denote by $\mathsf{brick}\, A$ the set of isomorphism classes of bricks of A, and by $\mathsf{sbrick}\, A$ the set of isomorphism classes of semibricks.

For a set $\mathscr{X}$ of A-modules, we denote by $\mathsf{T}(\mathscr{X})$ the smallest torsion class containing $\mathscr{X}$. We denote $\mathsf{f\text{-}brick}\, A \subseteq \mathsf{brick}\, A$ the set of bricks S such that $\mathsf{T}(S)$ is functorially finite. A first result about bricks, proved with Iyama and Jasso is the following one:

Theorem 3.3 ([DIJ, Theorem 4.1])**.** *Let A be a finite dimensional algebra. Then there is a bijection*

$$\mathsf{i}\tau\text{-}\mathsf{rigid}\, A \to \mathsf{f\text{-}brick}\, A$$

given by $X \mapsto X/\,\mathsf{rad}_E\, X$ for $E := \mathsf{End}_A(X)$.

As an application of Theorem 3.3, we give the following alternative characterization of τ-tilting finite algebras, which complements Theorem 3.1.

Theorem 3.4 ([DIJ, Theorem 4.2])**.** *Let A be a finite dimensional algebra. Then, the following conditions are equivalent.*

(i) *The algebra A is τ-tilting finite.*
(ii) *The set $\mathsf{brick}\, A$ is finite.*
(iii) *The set $\mathsf{f\text{-}brick}\, A$ is finite.*

We now suppose that A is τ-tilting finite. By Theorem 3.1, all torsion classes of $\mathsf{mod}\, A$ are functorially finite, hence $\mathsf{f\text{-}brick}\, A = \mathsf{brick}\, A$. We define a map

$$\mathsf{Hasse}_1(\mathsf{tors}\, A) = \mathsf{Hasse}_1(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A) \to \mathsf{brick}\, A$$

called the *brick labelling*. Let Rad_A be the Jacobson radical of $\mathsf{mod}\, A$. Consider an arrow $q : T \to U$ of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$. Then there are decompositions $T = X \oplus M$ and $U = Y \oplus M$ for indecomposable A-modules X and Y and we prove that

$$S_q := \frac{X}{\mathsf{Rad}_A(T, X) \cdot T}$$

is a brick, as $\mathsf{Rad}_A(T, X) \cdot T \in \mathsf{Fac}\, T$, hence $\mathsf{Ext}^1_A(X, \mathsf{Rad}_A(T, X) \cdot T) = 0$.

Definition 3.5. We call S_q the *label* of the arrow q.

Example 3.6. Consider the algebra

$$\Lambda := \Bbbk \left(1 \xrightarrow{\alpha} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3 \right) \Big/ (\alpha\beta, \beta\beta^*, \beta^*\beta).$$

We depict $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, \Lambda)$ in Figure 3.9. Labels of arrows are circled.

It turns out that the brick labelling categorifies the forcing order in the following sense. For $\mathcal{S} \in \mathsf{sbrick}\, A$, denote by $\mathsf{Filt}\, \mathcal{S}$ the full subcategory of $\mathsf{mod}\, A$ consisting of objects filtered by elements of $\mathcal{S}$. It is a *wide subcategory* of $\mathsf{mod}\, A$, *i.e.* it is closed under kernels, cokernels and extensions.

Theorem 3.7 ([DIR$^+$])**.** *Let q and q' be two arrows of $\mathsf{Hasse}(\mathsf{tors}\, A)$.*

(a) We have $q \leftrightsquigarrow q'$ if and only if $S_q = S_{q'}$.
(b) The forcing relation is the transitive closure of the following:

$$q \rightsquigarrow q' \text{ if } \exists(\{S_q\} \sqcup \mathcal{S}) \in \mathsf{sbrick}\, A, S_{q'} \in \mathsf{Filt}(\{S_q\} \sqcup \mathcal{S}) \setminus \mathsf{Filt}\, \mathcal{S}.$$

Thanks to Theorem 3.7(a), we consider $\rightsquigarrow$ as a partial order on $\mathsf{brick}\, A$, also denoted by $\rightsquigarrow$.

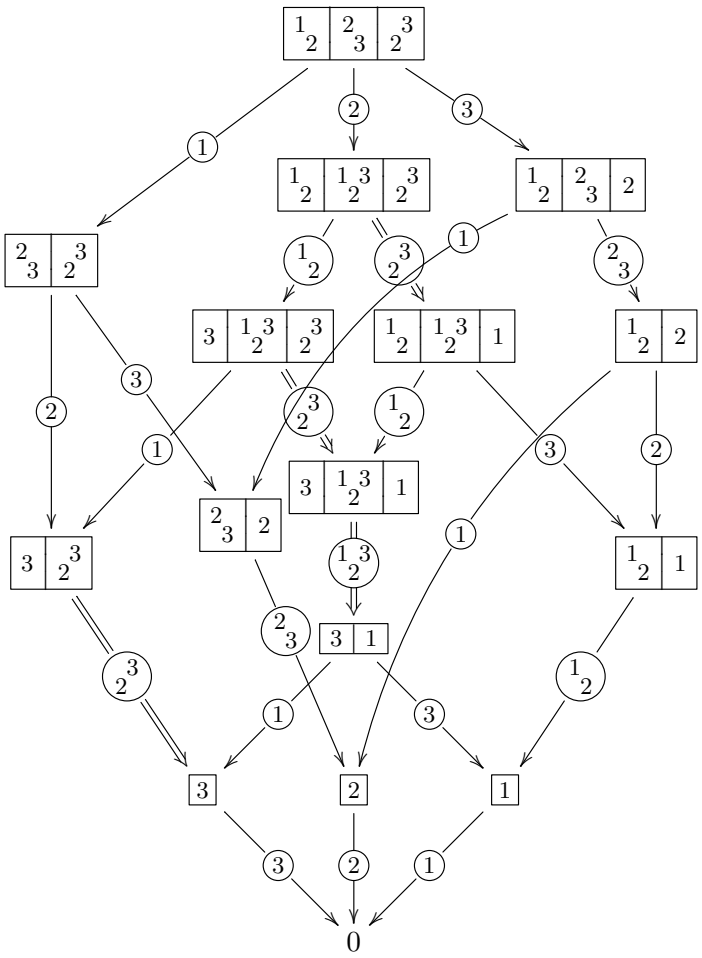


FIGURE 3.9. Hasse quiver of the lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$

Example 3.8. In Figure 3.9, the arrows labelled by bricks that are forced by ${}_2^3$ are doubled.

Using Theorem 3.7 with Theorem 3.3, we get the following important lattice theoretical result. Recall that $\mathscr{T} \in \mathsf{tors}\,A$ is *join-irreducible* if it is not 0 and it cannot be written non-trivially as the join of two other torsion classes. Equivalently, $\mathscr{T}$ is join-irreducible if there is a unique arrow pointing from $\mathscr{T}$ in $\mathsf{Hasse}(\mathsf{tors}\,A)$. Dually, we define the notion of *meet-irreducible* torsion classes.

Theorem 3.10 ([DIR$^+$])**.** *The lattice* $\mathsf{tors}\,A$ *is* congruence uniform*. In other terms, there are bijections:*

- *From join-irreducible torsion classes to forcing-equivalence classes of arrows of* $\mathsf{Hasse}(\mathsf{tors}\,A)$ *mapping* $\mathscr{T}$ *to the class of the unique arrow pointing from* $\mathscr{T}$*;*
- *From meet-irreducible torsion classes to forcing-equivalence classes of arrows of* $\mathsf{Hasse}(\mathsf{tors}\,A)$ *mapping* $\mathscr{T}$ *to the class of the unique arrow pointing toward* $\mathscr{T}$*.*

The argument for Theorem 3.10 is an easy consequence of the following facts (and their duals): First of all, join-irreducible torsion classes are exactly the ones of the form $\mathsf{Fac}\,X$ for $X \in \mathsf{i}\tau\text{-}\mathsf{rigid}\,A$. Secondly, the label of the unique arrow pointing from $\mathsf{Fac}\,X$ in this case is $X/\,\mathsf{rad}_E\,X$ for $E = \mathsf{End}_A(X)$. Then we conclude by Theorem 3.3 and Theorem 3.7. Notice that the surjectivity of both maps described in Theorem 3.10 is a general property of finite lattices, so the non-trivial point is the injectivity. It is important as in this case the forcing relation induces partial orders on the set of join-irreducible elements and on the set of meet-irreducible elements.

Another, more indirect, consequence of Theorem 3.10 is

Theorem 3.11 ([DIR$^+$]). (a) *For all* $(N, Q) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$*, the subcategory*

$$\mathscr{W}(N, Q) := {}^{\perp}(\tau N) \cap Q^{\perp} \cap N^{\perp}$$

is a wide subcategory of $\operatorname{mod} A$*.*

(b) *There is a bijection from* $\tau\text{-}\mathsf{tilt\text{-}pair}\, A$ *to the set of wide subcategories of* A*, mapping a pair* (T, P) *to* $\mathscr{W}(T/Y, P)$ *where* Y *is the minimal summand of* T *satisfying* $\mathsf{Fac}\, Y = \mathsf{Fac}\, T$*.*

The end of this subsection is devoted to generalize a characterization of the forcing order given in [IRRT] for preprojective algebras of Dynkin type, namely the *doubleton extension order*. Recall that $\Bbbk$ is a fixed base field, not necessarily algebraically closed.

Definition 3.12 ([IRRT]). The *doubleton extension order* on $\mathsf{brick}\, A$ is the transitive closure $\rightsquigarrow_{\mathsf{d}}$ of the relation defined by: $S_1 \rightsquigarrow_{\mathsf{d}} S_2$ if there exists a brick S_1' such that

$\dim \mathsf{Ext}^1_A(S_1, S_1') = 1$ and there is an exact sequence $0 \to S_1' \to S_2 \to S_1 \to 0$;
or $\dim \mathsf{Ext}^1_A(S_1', S_1) = 1$ and there is an exact sequence $0 \to S_1 \to S_2 \to S_1' \to 0$.

Notice that we do not assume that $\{S_1, S_1'\}$ is a semibrick, but it is automatic. We consider brick having the following stronger property.

Definition 3.13. A brick $S \in \operatorname{mod} A$ is called a *stone* if $\mathsf{Ext}^1_A(S, S) = 0$. It is called a *strong stone* if additionally $\mathsf{End}_A(S) \cong \Bbbk$.

We give the following alternative characterization of the forcing order.

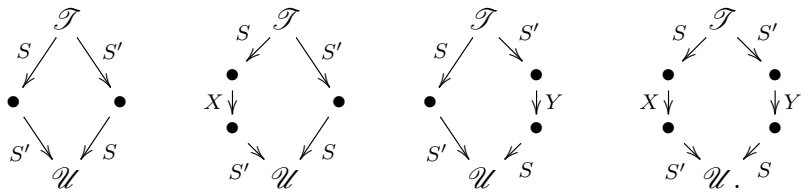
Theorem 3.14. *Let* A *be a finite dimension* $\Bbbk$*-algebra that is* τ*-tilting finite such that all bricks of* $\operatorname{mod} A$ *are strong stones. Then the forcing order* $\rightsquigarrow$ *on* $\mathsf{brick}\, A$ *coincides with the doubleton extension order* $\rightsquigarrow_{\mathsf{d}}$*.*

Notice that assumptions of Theorem 3.14 are often satisfied, for example for quotients of path algebras of Dynkin quivers, for preprojective algebras of Dynkin type, ...

Now, we suppose that A is τ-tilting finite and all bricks are strong stones. From the above, we deduce the following characterization of polygons in $\mathsf{tors}\, A$:

Proposition 3.15. *Suppose that* $[\mathscr{U}, \mathscr{T}]$ *is a polygon of* $\mathsf{tors}\, A$*.*

(a) *The polygon* $[\mathscr{U}, \mathscr{T}]$ *is labelled in one of the following ways:*



(b) *The label* X *appears (case* 2 *and* 4*) if and only if* $\mathsf{Ext}^1_A(S', S) \neq 0$*. If it is the case,* $\dim \mathsf{Ext}^1_A(S', S) = 1$ *and there is a non-split extension* $0 \to S \to X \to S' \to 0$*.*

(c) *The symmetric statement holds for* Y*.*

3.3. **Algebraic quotients of the lattice of torsion classes.** Recall that if A is a finite dimensional $\Bbbk$-algebra, a lattice quotient $\mathsf{tors}\, A \twoheadrightarrow L$ is *algebraic* if it is of the form $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B$ for a quotient $B = A/I$. This subsection is devoted to understand these algebraic quotients from the point of view of the brick labelling.

Let us fix a quotient $B = A/I$. A first observation about algebraic quotients is that their understanding at the level of support τ-tilting modules is well-behaved:

Proposition 3.16 ([DIR$^+$]). (a) *If* $X \in \tau\text{-}\mathsf{rigid}\, A$ *then* $B \otimes_A X \in \tau\text{-}\mathsf{rigid}\, B$*.*

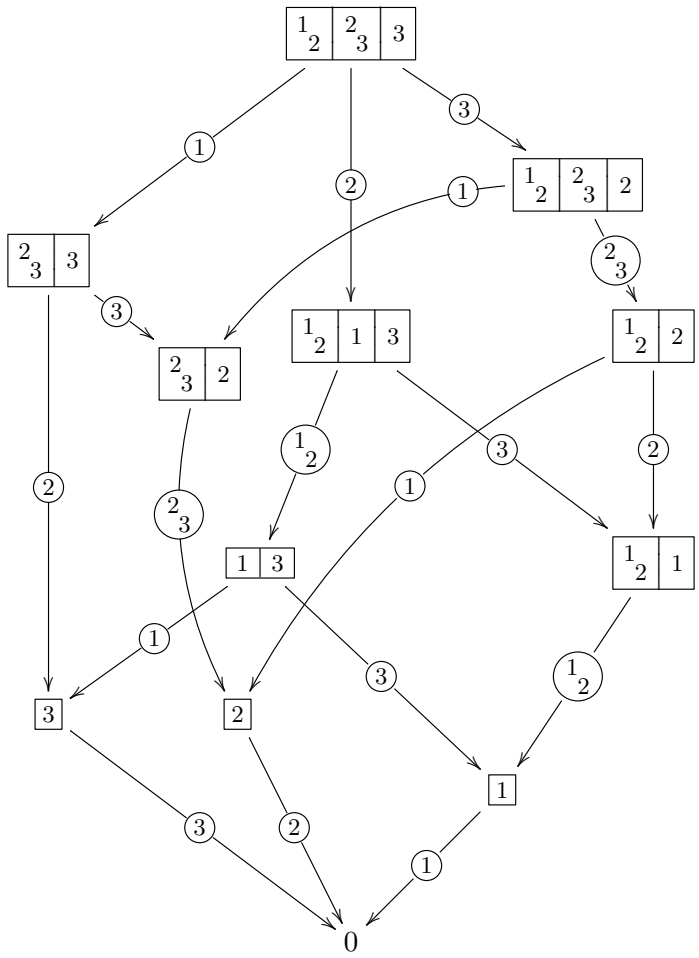


FIGURE 3.17. Hasse quiver of the lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda'$

(b) *We have a commutative diagram*

$$\begin{tikzcd}
\tau\text{-}\mathsf{rigid}\,A \arrow[r, "\mathsf{Fac}\,-"] \arrow[d, "B\otimes_A -"'] & \mathsf{tors}\,A \arrow[d, "\mathscr{T}\mapsto\mathscr{T}\cap\mathrm{mod}\,B"] \\
\tau\text{-}\mathsf{rigid}\,B \arrow[r, "\mathsf{Fac}\,-"] & \mathsf{tors}\,B.
\end{tikzcd}$$

Notice that the vertical arrow $B \otimes_A -$ of Proposition 3.16 is not necessarily surjective in general. However, in the case where A is τ-tilting finite, horizontal arrows are isomorphisms of lattices and both vertical arrows are surjective.

The main result concerning algebraic quotient from the point of view of brick labelling is the following one:

Theorem 3.18 ([DIR$^+$])**.** *Let A be a finite-dimensional $\Bbbk$-algebra that is τ-tilting finite, and I be an ideal of A. Then an arrow of* $\mathsf{Hasse}(\mathsf{tors}\,A)$ *is not contracted by Θ_I if and only if its label is in* $\mathrm{mod}(A/I)$*. Moreover, in this case, it has the same label in* $\mathsf{Hasse}(\mathsf{tors}\,A)$ *and* $\mathsf{Hasse}(\mathsf{tors}(A/I))$*.*

Example 3.19. We illustrate Theorem 3.18 by continuing Example 3.6. Let $\Lambda' := \Lambda/(\beta^*)$. Bricks that are not in $\mathrm{mod}(\Lambda')$ are the ones that have been doubled in Figure 3.9. We provide $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}(\Lambda'))$ in Figure 3.17, endowed with brick labelling to check Theorem 3.18. Notice that the same process can be applied to get Figure 3.17 from Figure 2.17, as Λ' is also a quotient of $\Bbbk(1 \to 2 \to 3)$.

We get the following corollary of Theorem 3.18.

Corollary 3.20 ([DIR$^+$]). *Let A be a finite-dimensional $\Bbbk$-algebra that is τ-tilting finite and I be an ideal of A. Then the following are equivalent:*

(i) $I \subseteq I_0 := \bigcap_{S \in \mathsf{brick}\, A} \mathsf{ann}\, S$ *where* $\mathsf{ann}\, S := \{a \in A \mid aS = 0\}$*;*
(ii) *The map* $\mathscr{T} \mapsto \mathscr{T} \cap \operatorname{mod}(A/I)$ *is an isomorphism from* $\mathsf{tors}\, A$ *to* $\mathsf{tors}(A/I)$*.*

In particular, I_0 is the maximum ideal of A satisfying each of these properties.

It permits to recover easily the following result by [EJR] in the τ-tilting finite case.

Corollary 3.21. *Let A be a finite dimensional $\Bbbk$-algebra that is τ-tilting finite and Z the center of A. Then for any $I \subset A\, \mathsf{rad}\, Z$, $\eta_A(I)$ is the trivial congruence.*

We now give a lattice theoretical insight about the application that maps an ideal I to the lattice quotient $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}(A/I)$. We denote by $\mathsf{Ideals}\, A$ the lattice of ideals of A. We denote by $\mathsf{Con}(\mathsf{tors}\, A)$ the lattice of congruence of $\mathsf{tors}\, A$, *i.e.* of equivalence relations respecting joins and meets (they are exactly kernels of lattice quotients). A congruence Θ is bigger than a congruence Ξ if all arrows contracted by Ξ are also contracted by Θ. We have the following general results:

Proposition 3.22 ([DIR$^+$]). *Let A be a finite dimensional algebra that is τ-tilting finite. The map $\eta_A : \mathsf{Ideals}\, A \to \mathsf{Con}(\mathsf{tors}\, A)$ mapping $I \in \mathsf{Ideals}\, A$ to the kernel Θ_I of $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}(A/I)$ is order preserving and commutes with joins (but not with meets in general).*

Remark that Proposition 3.22 generalizes to non-τ-tilting finite algebras by considering complete congruences, *i.e.* congruence compatible with infinite meets and joins. In this case η_A commutes also with infinite joins.

We finally give a necessary condition in terms of bricks for a congruence to be algebraic. Let us first introduce a combinatorial concept:

Definition 3.23. We say that a brick in $\operatorname{mod} A$ is

- *double join-irreducible* if it labels an arrow $\mathscr{T} \to \mathscr{U}$ with $\mathscr{T}$ join-irreducible and $\mathscr{U}$ join-irreducible or 0;
- *double meet-irreducible* if it labels an arrow $\mathscr{T} \to \mathscr{U}$ with $\mathscr{U}$ meet-irreducible and $\mathscr{T}$ meet-irreducible or $\operatorname{mod} A$;
- *small* if it is double join-irreducible and double meet-irreducible.

We get:

Theorem 3.24 ([DIR$^+$] for preprojective algebras, [DIRT] for the general case). *Let A be a τ-tilting finite $\Bbbk$-algebra, and $\pi : \mathsf{tors}\, A \twoheadrightarrow L$ be a lattice quotient. Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii) hold.*

(i) *π is an algebraic quotient.*
(ii) *L is Hasse-regular.*
(iii) *The set of labels of arrows contracted by π is the closure by forcing of a set of small bricks.*

Recall that A is *Schurian* if $\dim \mathsf{Hom}_A(P, P') \leq 1$ for any indecomposable projective modules P and P'. We get a converse ofTheorem 3.24, in good cases:

Theorem 3.25 ([DIR$^+$] for (a), [DIP] for (b) and (c)). *Suppose that we are in one of the following cases:*

(a) A is a preprojective algebra of type A;
(b) All indecomposable $X \in \operatorname{mod} A$ satisfy $\mathsf{End}_A(X) = \Bbbk$;
(c) A is Schurian, biserial and τ-tilting finite.

Then (iii)$\Rightarrow$(i) hold in Theorem 3.24.

Notice that in Theorem 3.25, cases (b) or (c) both imply (a), by using Corollary 3.20. The following example shows that we cannot expect (iii)$\Rightarrow$(i) in general:

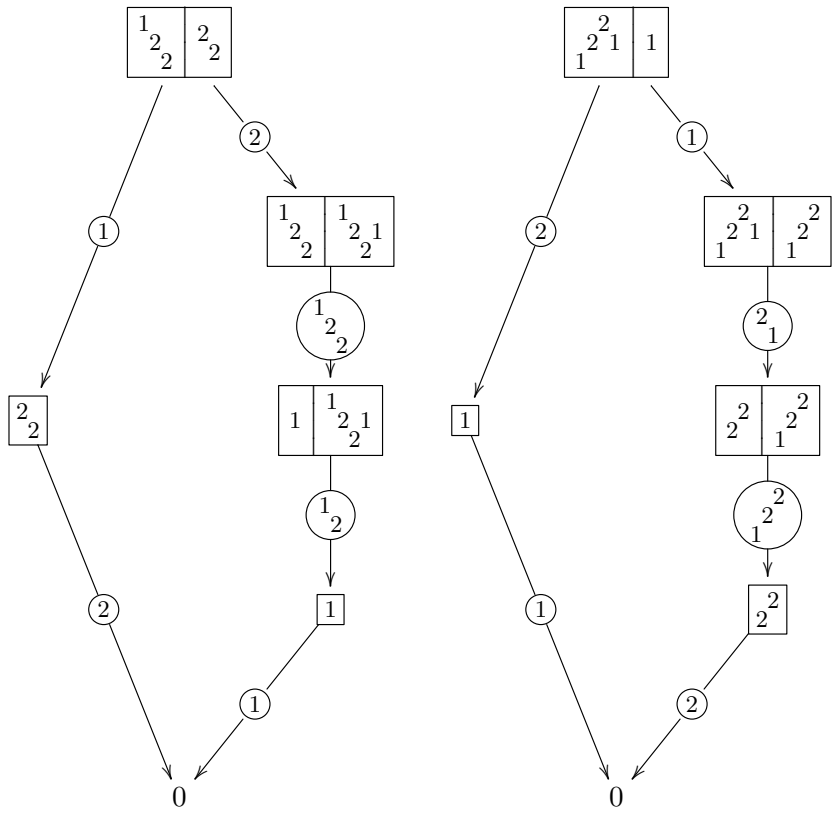


FIGURE 3.27. $\mathsf{Hasse}(\mathsf{tors}\,\Lambda)$ and $\mathsf{Hasse}(\mathsf{tors}\,\Lambda')$

Example 3.26. We consider the algebras

$$\Lambda := \frac{\Bbbk\left(1 \xrightarrow[\alpha]{} 2 \circlearrowleft \beta\right)}{(\beta^2)} \quad \text{and} \quad \Lambda' := \frac{\Bbbk\left(1 \xleftarrow[\alpha]{} 2 \circlearrowleft \beta\right)}{(\beta^2)}.$$

We depict the labelled Hasse quivers of their support τ-tilting modules in Figure 3.27. We observe that in the first case, algebraic quotients that do not contract S_1 or S_2 contract

$$\text{either only } {}^{1}{}_{2}{}_{2} \quad \text{or} \quad {}^{1}{}_{2} \text{ and } {}^{1}{}_{2}{}_{2}\,.$$

In the second case, they contract

$$\text{either only } {}_{1}{}^{2}{}^{2} \quad \text{or} \quad {}^{2}{}_{1} \text{ and } {}_{1}{}^{2}{}^{2}\,.$$

In particular, even though the Hasse quivers are isomorphic, the set of algebraic quotients are different. A third example is the hereditary algebra of type B_2 (on a non-algebraically closed field), which has the same Hasse quiver, while the two vertical are both contracted by any non-trivial algebraic congruence.

In other terms, we see that (ii) does not implies (i) in general. Furthermore, we see that in general algebraic quotients cannot be understood knowing only the Hasse quiver.

We now give the key step toward Theorem 3.25. Then we have

Theorem 3.28 ([DIP]). *Suppose that A is Schurian and τ-tilting finite. Then the map η_A of Proposition* 3.22 *is actually an isomorphism.*

3.4. **Application to preprojective algebras and Cambrian lattices.** In this subsection, we show how to reinterpret some known results about preprojective algebras and some lattices in the light of previous subsections.

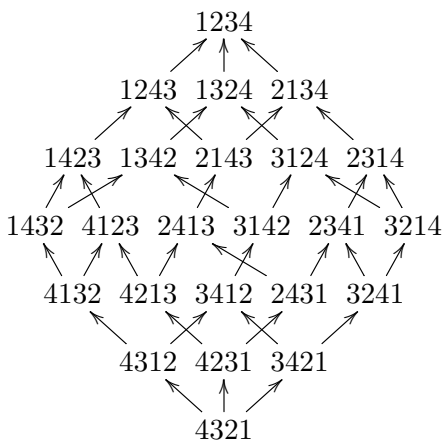


FIGURE 3.29. The weak order on $\mathfrak{S}_4$

Let Δ be a Dynkin graph, that is one of the following simply laced diagram:

$$A_n \qquad 1 - 2 - 3 - \cdots - (n-2) - (n-1) - n$$

$$D_n \qquad \begin{array}{c} n \\ | \\ 1 - 2 - 3 - \cdots - (n-2) - (n-1) \end{array}$$

$$E_n \qquad \begin{array}{c} n \\ | \\ 1 - 2 - 3 - \cdots - (n-3) - (n-2) - (n-1), \end{array} \qquad 6 \leq n \leq 8$$

It determines a group called the *Weyl group* W of Δ. Let $S := \{s_1, \ldots, s_n\}$. The Weyl group W of Δ is the group given by the presentation

$$W = \langle S \mid \forall i, s_i^2 = 1; \forall (i,j) \in \Delta_1, s_i s_j s_i = s_j s_i s_j; \forall (i,j) \notin \Delta_1, s_i s_j = s_j s_i \rangle,$$

where we write $(i,j) \in \Delta_1$ when i and j are adjacent in Δ. The best known example of a Weyl group is the symmetric group $\mathfrak{S}_{n+1}$, which is the Weyl group associated to a diagram of type A_n. The generators $s_1, \ldots, s_n$ are the simple transpositions $(1,2)$ through $(n, n+1)$.

We call an expression $s_{i_1} \cdots s_{i_k}$ a *word* for $w \in W$ if $w = s_{i_1} \cdots s_{i_k}$ holds in W. The minimal length (number of letters) of a word for w is called the *length* of w and denoted $\ell(w)$. A word for w having exactly $\ell(w)$ letters is called a *reduced word* for w.

The *(right) weak order* on W is the partial order on W setting $v \leq w$ if and only if there exists a reduced word $s_{i_1} \cdots s_{i_k}$ for w such that, for some $j \leq k$, the word $s_{i_1} \cdots s_{i_j}$ is a reduced word for v. Importantly for our purposes, the weak order on W is a lattice (see, for example, [BB1, Theorem 3.2.1]). Arrows of $\mathsf{Hasse}\, W$ are of the form $ws \to w$ for $s \in S$ whenever $\ell(ws) > \ell(w)$.

Example 3.30. We describe the weak order on permutations. An *inversion* of $\sigma \in \mathfrak{S}_{n+1}$ is a pair $1 \leq i < j \leq n+1$ such that $\sigma(i) > \sigma(j)$. The length of σ is the number of inversions of σ. The weak order on $\mathfrak{S}_{n+1}$ corresponds to containment of inversion sets. Hasse arrows $\tau \to \sigma$ are obtained from σ by swapping two adjacent entries $\sigma(i) < \sigma(i+1)$. We illustrate the weak order on $\mathfrak{S}_4$ in Figure 3.29, where permutation are given in *one-line notation*, that is $\sigma(1)\sigma(2)\cdots\sigma(n+1)$.

Let us now consider a quiver Q that is an orientation of our graph Δ. We define a new quiver $\overline{Q}$ by adding a new arrow $q^* : j \to i$ for each arrow $q : i \to j$ in Q. The *preprojective algebra* of Q is defined as

$$\Pi = \Pi_Q := \Bbbk \overline{Q} \Big/ \left(\sum_{a \in Q_1} (aa^* - a^* a) \right)$$

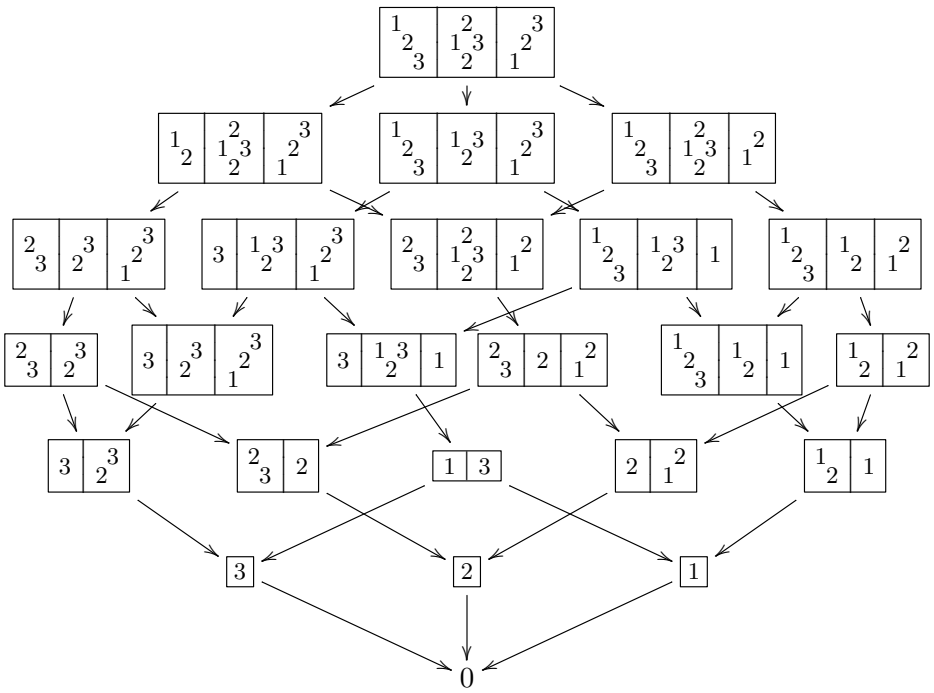


FIGURE 3.32. $\mathsf{s}\tau\text{-}\mathsf{tilt}(\Pi)$ in type A_3

Then, up to isomorphism, Π does not depend on the choice of orientation of the quiver Q, but only on Δ. As Δ is a Dynkin diagram, Π is finite-dimensional.

For a vertex $i \in Q_0$, we denote by e_i the corresponding idempotent of Π. We denote by I_i the two-sided ideal of Π generated by the idempotent $1 - e_i$. Then I_i is a maximal ideal of Π. For each element $w \in W$, we take a reduced word $w = s_{i_1} \cdots s_{i_k}$ for w, and let $I_w := I_{i_1} \cdots I_{i_k}$ which in fact depends only on w. The following result due to Mizuno is the starting point of this subsection.

Theorem 3.31 ([Miz, §2.14, §2.21])**.** *We have isomorphisms of lattices*

$$(W, \leq^{\mathsf{op}}) \xrightarrow{\sim} (\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Pi, \leq) \xrightarrow{\sim} (\mathsf{tors}\,\Pi, \subseteq).$$

given by $w \mapsto I_w \mapsto \mathsf{Fac}\, I_w$.

Example 3.33. In type A_3, the weak order on $W = \mathfrak{S}_4$ is displayed in Figure 3.29. The corresponding support τ-tilting modules are shown in Figure 3.32.

We use these results to re-derive the known connection between hereditary algebras of Dynkin type and Cambrian lattices. Proofs given in [DIR$^+$] bypass the combinatorics of sortable elements, which is needed in the previously known proofs.

A *Coxeter element* of W is an element c obtained as the product in any order of the generators $S = \{s_1, \ldots, s_n\}$. The quiver Q defines a Coxeter element given by an expression $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ such that if $i \leftarrow j$ then s_i appears before s_j in the expression $s_{i_1} s_{i_2} \cdots s_{i_n}$. There may be several expressions having this property, but they all define the same Coxeter element because if i and j are not related by an arrow of Q, the generators s_i and s_j commute. Conversely a Coxeter element c uniquely determines an orientation of the Dynkin diagram.

We use Q (or equivalently c) to define a lattice congruence Θ_c on W called the *c-Cambrian congruence*. We consider the set $\mathscr{E}_c := \{s_i s_j \to s_i \mid i \to j \in Q_1\}$ of arrows of $\mathsf{Hasse}\, W$ and we consider the congruence Θ_c generated by $\mathscr{E}_c$ via forcing.

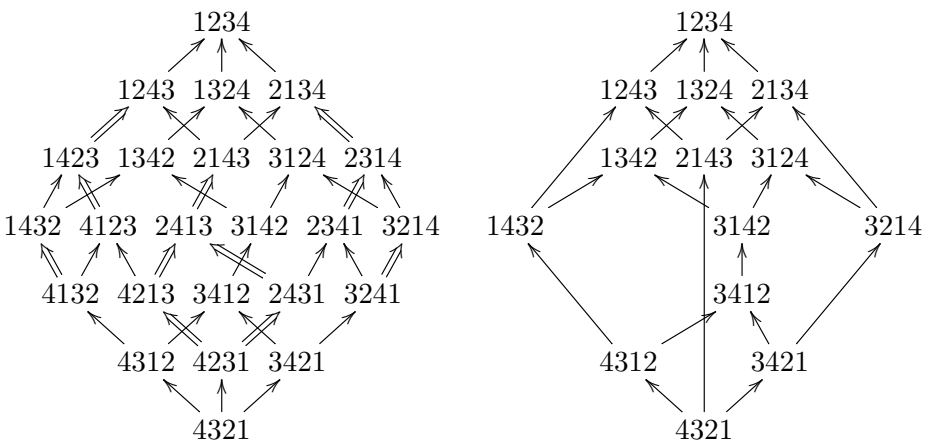


FIGURE 3.36. A Cambrian congruence and Cambrian lattice

Example 3.34. The Cambrian congruence Θ_c is illustrated in the left picture of Figure 3.36 for $W = \mathfrak{S}_4$ and $c = s_2 s_1 s_3$. Thus the congruence shown is the smallest congruence on $\mathfrak{S}_4$ contracting the arrows $2314 \to 2134$ and $1423 \to 1243$.

The quotient W/Θ_c is called the *c-Cambrian lattice*. The Cambrian lattice corresponding to the Cambrian congruence in the left picture of Figure 3.36 is shown in the right picture of Figure 3.36. The connection between torsion classes and Cambrian lattices was established in [IT]:

Theorem 3.35 ([IT])**.** *Let Q be a quiver of simply-laced Dynkin type, and c the corresponding Coxeter element. Then $\mathsf{tors}\,\Bbbk Q$ is isomorphic to the c-Cambrian lattice.*

This theorem was proved by showing that $\mathsf{tors}\,\Bbbk Q$ is isomorphic to the sublattice of W consisting of so-called *c-sortable elements*. We now give a direct representation theoretical argument in Theorem 3.37. Let $\Pi = \Pi_Q$ and I be the ideal $(a^* \mid a \in Q_1)$ of Π. Then, we consider the canonical projection

$$\varphi : \Pi \to \Pi/I = \Bbbk Q.$$

Theorem 3.37 ([DIR$^+$])**.** *The congruences Θ_c and Θ_I of $W \cong \mathsf{tors}\,\Pi$ coincide. Thus, there is a lattice isomorphism $\mathsf{tors}\,\Bbbk Q \cong W/\Theta_c$ making the following square commute:*

$$\begin{tikzcd} \mathsf{tors}\,\Pi \arrow[r, "\overline{(-)}"] \arrow[d] & \mathsf{tors}\,\Bbbk Q \arrow[d, dotted] \\ W \arrow[r] & W/\Theta_c. \end{tikzcd}$$

We also obtain in [DIR$^+$] a new, representation theoretical proof of the following result:

Theorem 3.38 ([Rea, Theorem 1.2])**.** *Consider the lattice quotient $\pi : W \twoheadrightarrow W/\Theta_c$. The set*

$$W_c := \{\min \pi^{-1}(x) \mid x \in W/\Theta_c\}$$

which consists of c-sortable elements is a sublattice of W that is isomorphic to W/Θ_c.

Notice that for a general congruence, the set W_c defined in Theorem 3.38 would only be a *join-semilattice* of L, *i.e.* closed under joins but not necessarily under meets.

3.5. **Torsion classes and their g-vectors.** In this subsection, we investigate a question that is closely related to the lattice structure of torsion classes. Namely, we study the *fan* of **g**-vectors of functorially finite torsion classes over an algebra, or equivalently, thanks to Proposition 2.15, of τ-rigid modules. As before, A is a finite dimensional algebra over a field $\Bbbk$. We fix a family of orthogonal primitive idempotents $e_1, e_2, \ldots, e_n$ of A. Most of the discussion comes from [DIJ].

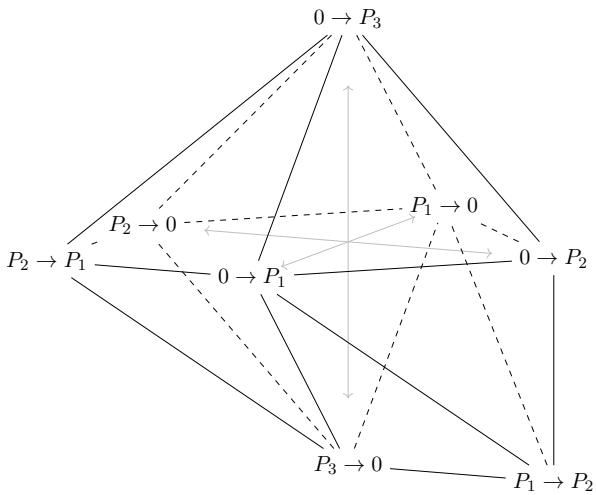


FIGURE 3.39. Geometric realization of $\Delta(\Lambda')$

Consider a τ-rigid A-module X. As $\operatorname{mod} A$ has enough projective objects, there is an exact sequence $P_X^1 \xrightarrow{u} P_X^0 \xrightarrow{v} X \to 0$ with u and v right minimal and P_X^1 and P_X^0 projective. Then $P_X^\bullet := (P_X^1 \to P_X^0)$ is uniquely determined (up to non-unique isomorphism) and is called the *minimal projective presentation of* X. Additionally, $P_X^\bullet$ is *presilting*, *i.e.* $\mathsf{Hom}_{\mathscr{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(P_X^\bullet, P_X^\bullet[n]) = 0$ for any $n > 0$ (we refer to classical textbooks, *e.g.* [Hap], for more details about the homotopy category $\mathscr{K}^{\mathrm{b}}(\mathsf{proj}\, A)$). More generally, if $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$, then its *minimal presentation*

$$P_{(X,P)} := P_X^\bullet \oplus P[1] = (P_X^1 \oplus P \to P_X^0)$$

is also presilting. If, additionally, (X, P) is a τ-tilting pair, then $P_X^\bullet \oplus P[1]$ is *silting*, *i.e.* maximal presilting up to multiplicities of direct summands.

We get in [DIJ, Proposition 6.3] that, for $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$, $P^0_{(X,P)}$ and $P^1_{(X,P)}$ have no common direct summands. Therefore, we define the $\mathbf{g}$-*vector* $\mathbf{g}_{(X,P)} = \mathbf{g} \in \mathbb{Z}^n$ in such a way that

$$P_{(X,P)} \cong \left(\bigoplus_{g_i<0} (Ae_i)^{-g_i} \to \bigoplus_{g_i>0} (Ae_i)^{g_i} \right).$$

Pushing the investigation further, we obtain that $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$ is uniquely determined by $\mathbf{g}_{(X,P)}$ up to isomorphism [DIJ, Proposition 6.5]. One of the reasons to investigate $\mathbf{g}$-vectors of τ-rigid modules is that they generalize the notion of $\mathbf{g}$-vectors of cluster-tilting objects in cluster categories in a natural way (see [DIJ, §6.4]). In particular, they also satisfy important properties like sign-coherence. Our point of view here is that it also enrich in a certain way the partial order structure of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$.

Consider $(T, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$. We denote $C(T, P)$ the cone generated by $\mathbf{g}$-vectors of direct summands of (T, P) in $\mathbb{R}^n$. Then we get:

Theorem 3.40 ([DIJ, Theorem 6.6])**.** *Let* $(T_1, P_1), (T_2, P_2) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$. *Then* $C(T_1, P_1) \cap C(T_2, P_2) = C(U, Q)$ *where* $(U, Q) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$ *satisfies* $\mathsf{add}(U, Q) = \mathsf{add}(T_1, P_1) \cap \mathsf{add}(T_2, P_2)$. *In particular,* $C(T_1, P_1)$ *and* $C(T_2, P_2)$ *intersect at their boundaries if* $\mathsf{add}(T_1, P_1) \neq \mathsf{add}(T_2, P_2)$.

Theorem 3.40 permits to define a geometric simplicial complex $\Delta(A)$ attached to A. This complex is dual to $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt\text{-}pair}\, A)$ (in the sense that vertices of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt\text{-}pair}\, A)$ correspond naturally to maximal n-cells of $\Delta(A)$ and arrows of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt\text{-}pair}\, A)$ to $(n-1)$ cells of $\Delta(A)$).

Example 3.41. As in Example 3.19, let Λ' be the algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ subject to the relation $\alpha\beta = 0$. The complex $\Delta(\Lambda')$ is illustrated in Figure 3.39. We replaced indecomposable τ-tilting pairs by their presentations.

A natural question arising from the previous discussion is the following one:

Question 3.42. *Is the partial order on τ-tilt-pair A entirely determined by the complex $\Delta(A)$?*

We give a partial answer to Question 3.42:

Theorem 3.43 ([DIJ, Theorem 6.11])**.** *Let $(X,P),(Y,Q) \in \tau$-tilt-pair A that are mutation of each other (*i.e. *neighbours in* Hasse(τ-tilt-pair A)*). Then the following conditions are equivalent, where $(L,R) \in \tau$-rigid-pair A is an object satisfying* $\mathsf{add}(X,P) \cap \mathsf{add}(Y,Q) = \mathsf{add}(L,R)$.

(i) $(X,P) > (Y,Q)$.
(ii) $\mathbb{R}^n_{\geq 0}$ *and $C(X,P)$ are contained in the same closed half-space defined by* span $C(L,R)$.
(iii) $\mathbb{R}^n_{\leq 0}$ *and $C(Y,Q)$ are contained in the same closed half-space defined by* span $C(L,R)$.

In other terms, $\Delta(A)$ determines entirely Hasse(τ-tilt-pair A). In particular, if A is τ-tilting finite, it determines the partial order on τ-tilt-pair A. We conjecture in [DIJ, Conjecture 6.14] that it is also true for τ-tilting infinite algebras.

We are also interested in the geometry of $\Delta(A)$. It is elementary to prove that if A is τ-tilting finite, then $\Delta(A)$ covers all $\mathbb{R}^n$ (see also [DIJ, §5]).

In [DIJ], we also prove that if A is τ-tilting finite, then $\Delta(A)$ has the so-called combinatorial property to be *shellable*. A consequence of this property is a strong analogous of simple connectedness. To give this property, let us start by a definition.

Definition 3.44. We say that a non-oriented cycle γ of Hasse(τ-tilt-pair A) has *rank at most ℓ* if there exists $(X,P) \in \tau$-rigid-pair A with $|A| - \ell$ indecomposable summands such that all vertices γ passes through have (X,P) as a direct summand.

Let γ and δ be two non-oriented cycles of Hasse(τ-tilt-pair A). We say that γ and δ are *related by cycles of rank ℓ in one step* if, up to cyclic rotation and change of orientation of γ and δ, we can write $\gamma = \varepsilon u$ and $\delta = \varepsilon v$ in such a way that uv^{-1} has rank at most ℓ. We say that γ and δ are *related by cycles of rank ℓ* if there exist a sequence $\gamma = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k = \delta$ such that for any $i = 1, 2, \ldots, k$, ε_i is related to ε_{i-1} by cycles of rank ℓ in one step.

Then we get:

Theorem 3.45 ([DIJ, Theorem 5.5])**.** *Suppose that A is τ-tilting finite. Then any non-oriented cycle of* Hasse(tors A) *is related by cycles of rank* 2 *to a trivial cycle.*

3.6. **Open problems.** Several natural questions are open in this section. We proved in [DIR$^+$] that, when A is τ-tilting finite, wide subcategories of mod A are parametrized by *polytopes* of tors A up to a certain equivalence investigated in [DIRT]. Here, we call polytope an interval $[\mathscr{U}, \mathscr{T}]$ of the partially ordered set tors A such that Hasse($[\mathscr{U}, \mathscr{T}]$) has constant valency at each vertex. As a consequence, we can compute the partially ordered set wide A of wide subcategories of mod A from Hasse(tors A). The converse is false. It leads to the following natural question:

Question 3.46. *Is there a natural combinatorial structure on* wide A *that makes the data of* tors A *and* wide A *equivalent?*

We expect a kind of duality, the two lattices having very different behaviour (for instance, tors A is Hasse regular, while maximal chains of wide A have constant length equal to the degree of regularity of tors A).

Question 3.42 is also a natural question we want to investigate about **g**-vectors.

On a longer perspective, we want to reply to the following question, probably much harder than the previous ones:

Question 3.47. *For a finite dimensional algebra A, is it true that any non-oriented cycle of* Hasse(tors A) *is related by cycles of rank* 2 *to a trivial cycle?*

It would generalize Theorem 3.45 and a positive answer would have important consequences which go far beyond the study of torsion classes. For example, Nakanishi [Nak] describes *dilogarithm identities* coming from cycles in the exchange graph of a cluster algebra. It turns out,

as explained in Section 5 that, as long as the cluster algebra can be categorified (and probably more often using alternative strategies), we can interpret these cycles as realized in a quiver $\mathsf{Hasse}(\mathsf{tors}\, A)$. Thus, Question 3.47 would imply that these dilogarithm identities come actually from elementary ones (coming from rank at most 2 cycles). This is known to be an important question in the study of dilogarithm identities.

About **g**-vectors, we ask the following two questions discussing the converse of an observation made at the end of previous section. The first one is probably elementary while the second one seems much harder:

Question 3.48. *Is it true that if $\Delta(A)$ covers $\mathbb{R}^n$, then A is τ-tilting finite?*

Question 3.49. *Is it true that if $\Delta(A)$ contains $\mathbb{Z}^n$, then A is τ-tilting finite?*

Notice also that new techniques have been recently elaborated by Gross, Hacking, Keel and Kontsevich [GHKK], which permit to reinterpret **g**-vectors as coming from more elaborated combinatorial-geometric objects, the so-called *scattering diagrams*, in cases coming from cluster algebras. More precisely, scattering diagrams extend in a certain way the previous complex to the whole ambient space. These very new techniques probably did not give yet all results we can expect from them. In particular, it is not clear if such a diagram exists for every finite dimensional algebra, which would be related to **g**-vectors of τ-rigid objects.

4. Categorification of cluster algebras via Cohen-Macaulay modules

The aim of this section is to use Cohen-Macaulay modules over certain orders to categorify some cluster algebras. A *cluster algebra* is a $\Bbbk$-algebra $\mathcal{A}$ endowed with a set of generators called *cluster variables*, grouped into sets of the same finite cardinality called *clusters*. Each cluster consists of algebraically independent elements. A finite number r of cluster variables are called *coefficients* and belong to all clusters. Finally, clusters are related by mutations. If $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ is a cluster and x_k is not a coefficient, then there exist one cluster $\mathbf{x} \setminus \{x_k\} \cup \{x_k^*\}$ called *mutation of* $\mathbf{x}$ *in direction* k satisfying $x_k x_k^* = M_1 + M_2$ where M_1 and M_2 are monomial functions of the set $\mathbf{x} \setminus \{x_k\}$, defined by a combinatorial rule in such a way that the mutation in direction k is an involution. The *exchange graph* $\Gamma_{\mathcal{A}}$ of $\mathcal{A}$ has set of vertices the set of clusters of $\mathcal{A}$ and a (non-oriented) edge between two cluster whenever they are mutations of each other.

Categorifying $\mathscr{A}$ consists of finding a categorical interpretation of $\Gamma_{\mathcal{A}}$ in a category. We will be interested here in *additive categorifications*. In this context, we consider a triangulated Krull-Schmidt $\Bbbk$-category $\mathscr{C}$ or an exact $\Bbbk$-category $\mathscr{E}$. We suppose that $\mathscr{C}$ is 2-Calabi-Yau or $\mathscr{E}$ is Frobenius stably 2-Calabi-Yau. In the triangulated case, it means that $\mathsf{Hom}_{\mathscr{C}}(X, Y[1]) \cong \mathsf{D}\,\mathsf{Hom}_{\mathscr{C}}(Y, X[1])$ functorially in X and Y and in the exact case, it means that $\mathscr{E}$ has enough projective and enough injective objects and $\mathsf{Ext}^1_{\mathscr{E}}(X, Y) \cong \mathsf{D}\,\mathsf{Ext}^1_{\mathscr{E}}(Y, X)$ functorially in X and Y, hence, in particular, projective objects and injective objects of $\mathscr{E}$ coincide. It is equivalent to say that $\mathscr{E}$ is Frobenius and the stable category $\underline{\mathscr{E}}$ is 2-Calabi-Yau. We will make the additional assumption that $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ is finite dimensional in the triangulated case and $\mathsf{Ext}^1_{\mathscr{E}}(X, Y)$ is finite dimensional in the exact case (we do not make assumptions about $\mathsf{Hom}_{\mathscr{E}}(X, Y)$). We say that $T \in \mathscr{C}$ is *cluster-tilting* if

$$\mathsf{add}\, T = \{X \in \mathscr{C} \mid \mathsf{Hom}_{\mathscr{C}}(X, T[1]) = 0\}$$

and we say that $T \in \mathscr{E}$ is cluster-tilting if

$$\mathsf{add}\, T = \{X \in \mathscr{E} \mid \mathsf{Ext}^1_{\mathscr{E}}(X, T) = 0\}.$$

Notice that $T \in \mathscr{E}$ is *cluster-tilting* if and only if it contains all projective objects of $\mathscr{E}$ as direct summands and it is cluster-tilting in the stable category $\underline{\mathscr{E}}$. Consider $T \in \mathscr{C}$ or $T \in \mathscr{E}$ basic cluster-tilting (*i.e.* without repeated direct summand). Consider also T_k an indecomposable direct summand of T that is not projective in the exact case. Then, if all non-invertible endomorphisms of T_k factor through T/T_k, there exists a cluster-tilting object $\mu_{T_k}(T) := T/T_k \oplus T_k^*$ characterized by the existence of a triangle

$$\xi : T_k \xrightarrow{u} U \xrightarrow{v} T_k^* \to T_k[1]$$

in the triangulated case and a conflation

$$\xi : 0 \to T_k \xrightarrow{u} U \xrightarrow{v} T_k^* \to 0$$

in the exact case where u is a left minimal $\mathsf{add}(T/T_k)$-approximation of T_k. Then v is automatically a right minimal $\mathsf{add}(T/T_k)$-approximation. We call $\mu_{T_k}(T)$ the *mutation of* T *in direction* T_k. The 2-Calabi-Yau property ensures the existence of a triangle

$$\rho : T_k^* \xrightarrow{u'} V \xrightarrow{v'} T_k \to T_k^*[1]$$

in the triangulated case and a conflation

$$\rho : 0 \to T_k^* \xrightarrow{u'} V \xrightarrow{v'} T_k \to 0$$

in the exact case having analogous properties as ξ, so that $\mu_{T_k^*}(\mu_{T_k}(T)) \cong T$. We now make the assumption that we can iterate this process arbitrarily many times. We call *accessible cluster-tilting objects* all objects obtained by mutation in any sequence of directions from a given initial object T. We call *accessible rigid objects* their indecomposable direct summands. We say that $\mathscr{C}$ or $\mathscr{E}$ *categorify* $\mathcal{A}$ if there exist a bijection φ from the set of accessible rigid objects of $\mathscr{C}$ or

$\mathscr{E}$ up to isomorphism and the cluster variables of $\mathcal{A}$ in such a way that clusters correspond to cluster-tilting objects and the mutation in the cluster algebra has the form

$$\varphi(T_k)\varphi(T_k^*) = \varphi(U) + \varphi(V)$$

where U and V are the middle terms of ξ and ρ as before and we extend φ by $\varphi(X \oplus Y) = \varphi(X)\varphi(Y)$. In this situation φ is called a *cluster character*. Notice that categorification by triangulated categories always give cluster algebras without coefficients while categorification by exact categories give cluster algebras with coefficients corresponding to indecomposable projective objects. Notice that here, we do not make any assumption about the form of φ. It might be more or less explicit depending on cases. Note also that, usually, φ is given in a form that can be extended to whole $\mathscr{C}$ or $\mathscr{E}$, and certain properties of φ extend to the whole category as well.

Categorifications of cluster algebras, either by triangulated categories or by exact categories have been widely studied. Cite for example [BMR$^+$, Ami] for triangulated categories. For an algebra A with good property, including all path algebras of acyclic quiver, they produce a triangulated category $\mathscr{C}(A)$ called *cluster category* satisfying the assumptions required above and having an *initial* cluster-tilting object T such that $\mathsf{End}_{\mathscr{C}(A)}(T) \cong A$. Then a cluster character has been constructed in [CC, Pal]. It categorifies in particular all so-called *symmetric* cluster algebras having finitely many clusters. In Subsection 4.1, we consider two cases, namely cluster algebras of type A and D having particularly nice combinatorial models. In these models, clusters correspond to triangulations of a polygon or a once punctured polygon and the mutations correspond to *flips* of arcs. Then cluster variables are in correspondence with arcs. We define a Frobenius exact category whose stable category is the cluster category and indecomposable projective objects correspond to the sides of the polygon. It is very natural from a combinatorial point of view and permits meanwhile to introduce a family of particularly well-behaved orders over $\Bbbk[t]$, the Cohen-Macaulay over which can be completely studied combinatorially.

Another case of interest in this section is the categorification via exact categories of a cluster algebra structure over the multi-homogeneous coordinate ring of partial flag varieties of type A, D, E. This categorification has been introduced in [GLS2] and uses some full exact subcategories of the module categories over preprojective algebras. The categorification misses certain coefficients (so that it actually categorify a certain quotient of the multi-homogeneous coordinate ring). In Subsection 4.2, we introduce orders which permit to add the missing coefficients. We also give a general framework to add such coefficients in good cases.

4.1. **Ice quivers with potential associated with triangulations.** We present in this section results obtained with Xueyu Luo in [DL1, DL2]. Our aim is to enhance results about Jacobian algebras of triangulations of polygons with at most one puncture by considering ice quivers with potential and certain Frobenius categories given by Cohen-Macaulay modules. We enlarge the connection between Cohen-Macaulay representation theory and cluster categories by studying the frozen Jacobian algebras associated with triangulations of surfaces from the viewpoint of Cohen-Macaulay representation theory. Throughout, let $\Bbbk$ denote a field and $R = \Bbbk[t]$. We first extend the construction of Caldero–Chapoton–Schiffler [CCS].

Let P be a polygon with n vertices or a polygon with n vertices and one puncture and σ be a triangulation of P. We define a quiver Q_σ attached to this situation. The set of vertices of Q_σ consists of arcs of σ and sides of P and the arrows are winding counter-clockwisely around each vertex of P and around the puncture if it exist. We remove outside arrows winding around a vertex that is not incident to any arc in σ. See Figure 4.1 for an example. We attach also to σ a *potential* W_σ, that is a linear combination of oriented cycles of Q_σ. Namely, Q_σ is the sum of clockwise 3-cycles corresponding to the triangle minus the sum of anticlockwise cycle winding around vertices and around the puncture if it exists. In Figure 4.1, we get $W_\sigma = abc + def + ghi - \alpha beh - \beta dc - \gamma gf$. For an arrow a of Q_σ and a cycle $u_1u_2\cdots u_\ell$ of Q_σ, we denote

$$\partial_a(u_1u_2\cdots u_\ell) := \sum_{u_i=a} u_{i+1}u_{i+2}\cdots u_\ell u_1\cdots u_{i-1}$$

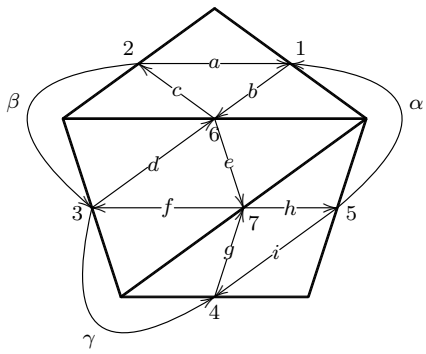


FIGURE 4.1. Quivers associated with a triangulation

and we extend this definition to potentials by linearity. Denote by $F \subset Q_{\sigma,0}$ the set of vertices corresponding to sides of the polygon. We call *frozen* these vertices. Following [BIRS], we define the *frozen Jacobian algebra of* σ to be

$$\Gamma_\sigma := \mathscr{P}(Q_\sigma, W_\sigma, F) := \Bbbk Q_\sigma / \mathscr{J}(W_\sigma, F),$$

where $\mathscr{J}(W_\sigma, F)$ is the ideal

$$\mathscr{J}(W_\sigma, F) = (\partial_a W_\sigma \mid a \in Q_{\sigma,1},\ s(a) \notin F \ \text{ or } \ t(a) \notin F)$$

of $\Bbbk Q_\sigma$. In the case of a punctured polygon, these definitions can be generalized to *tagged triangulations* as defined in [FST]. See [DL2] for the general definition. For example, in Figure 4.1, we have

$$\mathscr{J}(W_\sigma, F) = (ca - eh\alpha, ab - \beta d, ef - c\beta, fd - h\alpha b, de - \gamma g, hi - f\gamma, ig - \alpha be)$$

We get the following first result about Γ_σ:

Theorem 4.2 ([DL1, Theorem 1.1] and [DL2, Theorem 1.1])**.** *Let e_F be the sum of the idempotents of Γ_σ at frozen vertices. Then*

(a) *The frozen Jacobian algebra Γ_σ has the structure of an R-order,* i.e. *an R-module that is free of finite rank over R (see also Subsection 1.1).*
(b) *The R-order $e_F \Gamma_\sigma e_F$ is isomorphic to the Gorenstein order*

$$\textit{unpunctured case:} \quad \Lambda := \begin{bmatrix} R & R & R & \cdots & R & (t^{-1}) \\ (t) & R & R & \cdots & R & R \\ (t^2) & (t) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (t^2) & (t^2) & (t^2) & \cdots & R & R \\ (t^2) & (t^2) & (t^2) & \cdots & (t) & R \end{bmatrix}_{n \times n}$$

$$\textit{punctured case:} \quad \Lambda := \begin{bmatrix} R-R & R-R & R-R & \cdots & R-R & R \times R \\ (t) \times (t) & R-R & R-R & \cdots & R-R & R-R \\ (t)-(t) & (t) \times (t) & R-R & \cdots & R-R & R-R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (t)-(t) & (t)-(t) & (t)-(t) & \cdots & R-R & R-R \\ (t)-(t) & (t)-(t) & (t)-(t) & \cdots & (t) \times (t) & R-R \end{bmatrix}_{n \times n}$$

where $R - R := \{(P, Q) \in R^2 \mid P - Q \in (t)\}$ and $(t) - (t)$ is the ideal of $R - R$ generated by (t, t).

We get the following combinatorial understanding of the category $\mathsf{CM}\,\Lambda$ (see Section 1.1):

Theorem 4.3 ([DL1, Theorem 1.3] and [DL2, Theorem 1.4]). *The category* $\mathsf{CM}\,\Lambda$ *satisfies the following properties:*

(a) *For any tagged triangulation* σ *of* P*, we can map each tagged arc* a *of* σ *to the indecomposable Cohen-Macaulay* Λ*-module* $e_F\Gamma_\sigma e_a$*, where* e_a *is the idempotent of* Γ_σ *at* a*. This module does only depend on* a *(not on* σ*) and this map induces one-to-one correspondences*

$$\{\textit{sides and tagged arcs of } P\} \longleftrightarrow \{\textit{indecomposable objects of } \mathsf{CM}\,\Lambda\}\,/\cong$$
$$\{\textit{sides of } P\} \longleftrightarrow \{\textit{indecomposable projectives of } \mathsf{CM}\,\Lambda\}\,/\cong$$
$$\{\textit{tagged triangulations of } P\} \longleftrightarrow \{\textit{basic cluster-tilting objects of } \mathsf{CM}\,\Lambda\}\,/\cong .$$

(b) *For the cluster-tilting object* $T_\sigma := e_F\Gamma_\sigma$ *corresponding to a tagged triangulation* σ*,*

$$\mathsf{End}_{\mathsf{CM}\,\Lambda}(T_\sigma) \cong \Gamma_\sigma^{\mathrm{op}}.$$

(c) *The category* $\underline{\mathsf{CM}}\,\Lambda$ *is* 2*-Calabi-Yau.*

(d) *If* $\Bbbk$ *is a perfect field, there is a triangle-equivalence* $\mathscr{C}(\Bbbk Q) \cong \underline{\mathsf{CM}}\,\Lambda$*, where* Q *is a quiver of type* A_{n-3} *in the unpunctured case and* D_n *in the punctured case and* $\mathscr{C}(\Bbbk Q)$ *is the corresponding cluster category.*

In particular we get that $\mathsf{CM}\,\Lambda$ has the Krull-Schmidt property, even if $\mathsf{End}_{\mathsf{CM}\,\Lambda}(X)$ is not a local ring when X is indecomposable. This behaviour is rare and is worth to be noted. See Subsection 1.1 for more details. Also, $\mathsf{CM}\,\Lambda$ has an Auslander-Reiten theory, and the Auslander-Reiten quiver is illustrated in Figures 4.4, 4.5, 4.6 and 4.7 replacing indecomposable objects of $\mathsf{CM}\,\Lambda$ by the corresponding tagged arc via Theorem 4.3. Notice that in the punctured case, there are two tagged arcs going from a vertex i of the polygon to the puncture. These two tagged arcs are denoted by $(i,*)$ and $(i,\bowtie)$.

Theorem 4.3 permits to enhance the usual cluster category of type A_{n-3} or D_n with projective objects corresponding through the categorification to coefficients of the cluster algebra, in a way which fits perfectly with the combinatorial interpretation of tagged triangulations of P or P^*. Indeed, there is exactly one coefficient per side of the polygon and the situation is invariant by rotation of the polygon. More precisely, in the unpunctured case, using the cluster character X' defined in [FK, §3] on Frobenius categories (see also [Pal]), we get the following theorem.

Theorem 4.8 ([DL1, Theorem 1.4]). *If* $\Bbbk$ *is algebraically closed, in the unpunctured case, the category* $\mathsf{CM}(\Lambda)$ *categorifies through the cluster character* X' *a cluster algebra structure on the homogeneous coordinate ring of the Grassmannian of* 2*-dimensional planes in* $\mathbb{L}^n$ *(for any field* $\mathbb{L}$*). This cluster algebra structure coincides with the one defined in* [FZ1] *from Plücker coordinates.*

Similar results are obtained in the categorifications of the cluster structures of coordinate rings of (general) Grassmannians independently by Baur, King, Marsh [BKM] (see also Jensen, King, Su [JKS, Theorem 4.5]) and more generally for any partial flag variety in a recent work with Iyama (see Subsection 4.2).

Usually, the cluster category $\mathscr{C}(\Bbbk Q)$ is constructed as an orbit category of the bounded derived category $\mathscr{D}^{\mathrm{b}}(\Bbbk Q)$. We can reinterpret this result in this context by studying the category of Cohen-Macaulay graded Λ-modules $\mathsf{CM}^{\mathbb{Z}}(\Lambda)$:

Theorem 4.9 ([DL1, Theorem 1.5] and [DL2, Theorem 1.6]). *Using the same notation as before:*

(a) *The Cohen-Macaulay* Λ*-module* T_σ *can be lifted to a tilting object in* $\underline{\mathsf{CM}}^{\mathbb{Z}}(\Lambda)$*.*

(b) *There exists a triangle-equivalence* $\mathscr{D}^{\mathrm{b}}(\Bbbk Q) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(\Lambda)$*.*

Notice that the naive generalization of these results to other surfaces do not hold in general as shown in [DL2, §2.5] for a digon with two punctures. To be more explicit, for the cluster category $\mathscr{C}$ associated to a surface via [Ami] and [LF], a natural problem consists in finding an exact Frobenius category $\mathscr{E}$ satisfying $\underline{\mathscr{E}} \cong \mathscr{C}$. However, there is no reason to expect that projective-injective objects of $\mathscr{E}$ could be canonically indexed by some boundary arcs of the surface (for instance because there could be no boundary), or even by any other combinatorial

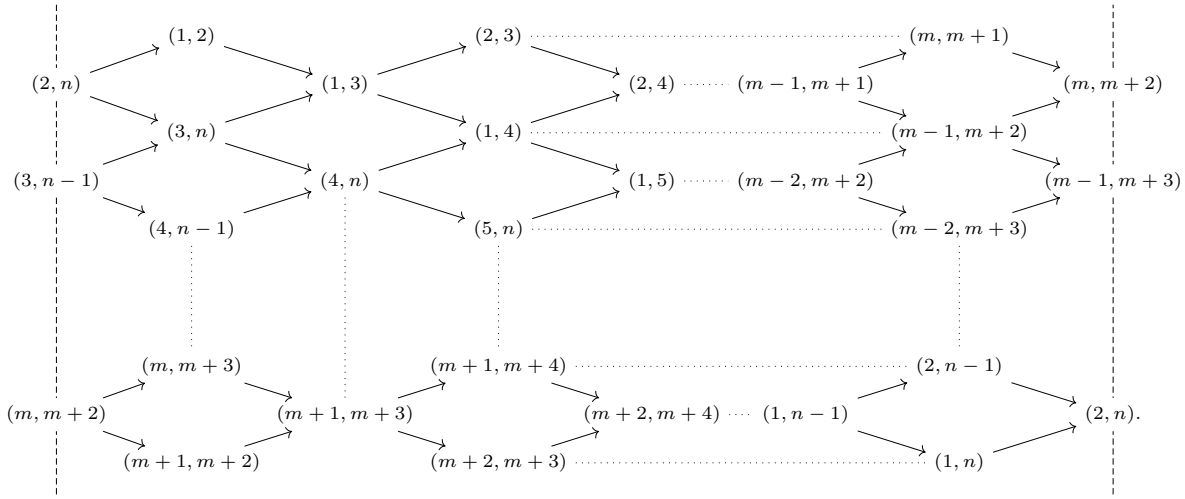


FIGURE 4.4. $\mathsf{CM}(\Lambda)$ for $n = 2m$ in the unpunctured case

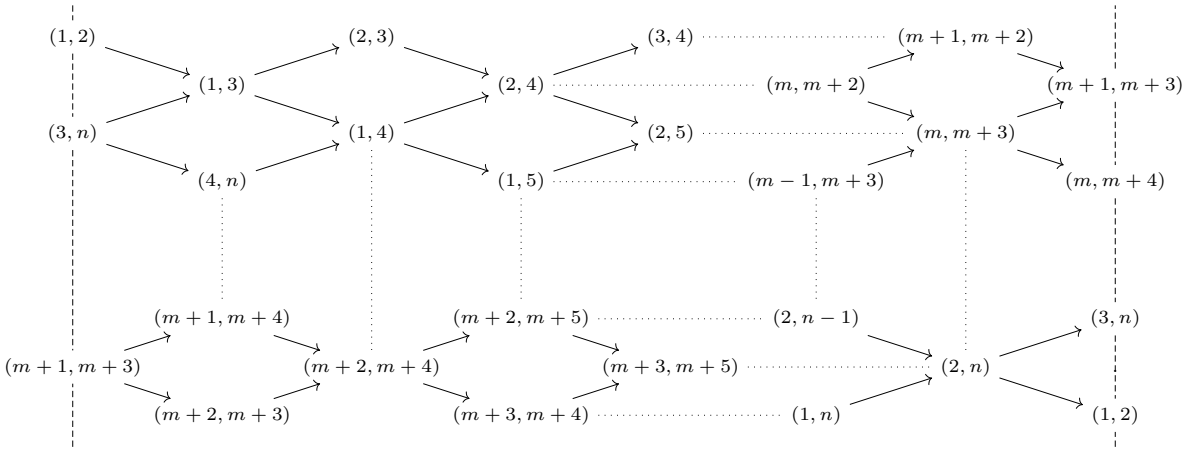


FIGURE 4.5. $\mathsf{CM}(\Lambda)$ for $n = 2m + 1$ in the unpunctured case

datum coming from the surface. Moreover, the natural generalization of Γ_σ does not behave nicely in general as shown in [Dem4, §3].

4.2. **Lifting preprojective algebras to orders and categorifying partial flag varieties.** In this subsection, we explain results of an article with Osamu Iyama [DI1]. In [GLS2], Geiss–Leclerc–Schröer introduced a cluster algebra structure on some subalgebra $\widetilde{\mathcal{A}}$ of the multi-homogeneous coordinate ring $\mathbb{C}[\mathscr{F}]$ of the partial flag variety $\mathscr{F} = \mathscr{F}(\Delta, J)$ corresponding to a Dynkin diagram Δ and a set J of vertices of Δ. They prove that $\widetilde{\mathcal{A}} = \mathbb{C}[\mathscr{F}]$ in type A, and conjecture that the equality holds after an appropriate localization for any Dynkin type. This structure generalizes previously known cases of Grassmannians, introduced from the beginning for $\mathsf{Gr}_2(\mathbb{C}^n)$ by Fomin and Zelevinsky [FZ2] (see also [BFZ]) and generalized by Scott for $\mathsf{Gr}_k(\mathbb{C}^n)$ [Sco].

In the same paper, Geiss–Leclerc–Schröer introduce a partial categorification of this cluster algebra structure on $\widetilde{\mathcal{A}}$. A crucial role is played by the preprojective algebra Π of type Δ and a certain full subcategory $\mathsf{Sub}\, Q_J$ of $\operatorname{mod}\Pi$ which is Frobenius and stably 2-Calabi-Yau. See Subsection 3.4 for the definition of Π. More precisely, they introduce a cluster character

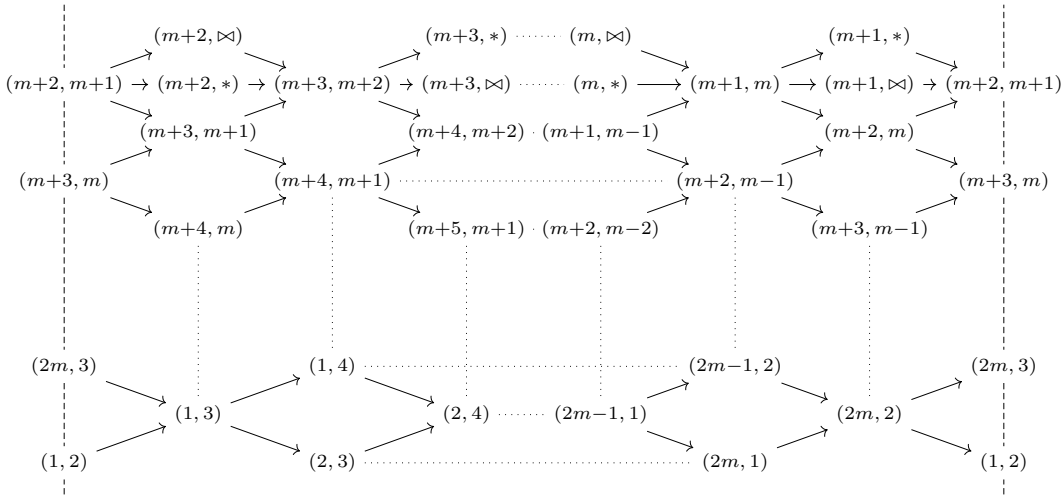


FIGURE 4.6. $\mathsf{CM}\,\Lambda$ for $n = 2m$ in the once-punctured case

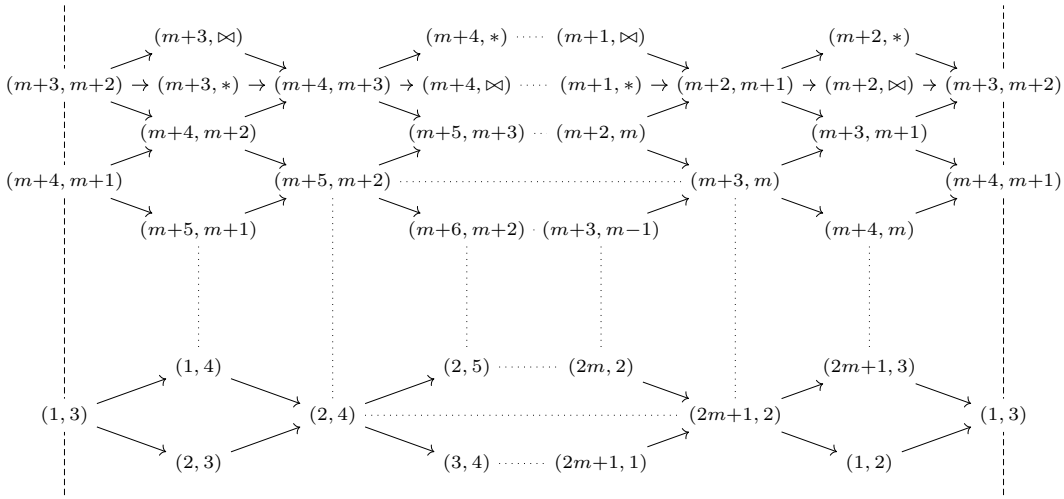


FIGURE 4.7. $\mathsf{CM}\,\Lambda$ for $n = 2m + 1$ in the once-punctured case

$\widetilde{\varphi} : \mathsf{Sub}\, Q_J \to \widetilde{\mathcal{A}}$ which gives a bijection

$$\begin{aligned} &\{\text{reachable indecomposable rigid objects in } \mathsf{Sub}\, Q_J\}/\cong \\ \xleftrightarrow{1-1} &\{\text{cluster variables and coefficients of } \widetilde{\mathcal{A}}\} \setminus \{\Delta_j \mid j \in J\}, \end{aligned}$$

where, for $j \in J$, Δ_j is the corresponding principal generalized minor.

One of the aim of this subsection is to look for a stably 2-Calabi-Yau category extending $\mathsf{Sub}\, Q_J$ whose reachable indecomposable rigid objects correspond to cluster variables and all coefficients of $\widetilde{\mathcal{A}}$. In [JKS], Jensen-King-Su achieved this in the case of classical Grassmannians (*i.e.* $\Delta = A_n$ for $n \geqslant 0$ and $\#J = 1$) by using orders (see also [BKM] for an interpretation in terms of dimer models). A special case is treated independently in Subsection 4.1. We extend their method to any arbitrary Dynkin diagram Δ and arbitrary set of vertices J.

For simplicity let $R := \Bbbk[\![t]\!]$ be the formal power series ring over an arbitrary field $\Bbbk$ (most results generalize to complete discrete valuation rings). For an idempotent $e \in A$, we define

$$\mathsf{CM}_e\, A := \{X \in \mathsf{CM}\, A \mid eX \in \mathsf{proj}(eAe)\}.$$

We prove the following result:

Theorem 4.10 ([DI1, Theorem A])**.** *Let Δ be a Dynkin diagram, and J be a set of vertices of Δ. Then, there exist a $\mathbb{C}[\![t]\!]$-order A, an idempotent $e \in A$ such that $\mathsf{CM}_e\, A$ is Frobenius and stably 2-Calabi-Yau, and a cluster character $\psi : \mathsf{CM}_e\, A \to \widetilde{\mathcal{A}}$ such that*

(a) *ψ induces a bijection between*
- *isomorphism classes of reachable indecomposable rigid objects of $\mathsf{CM}_e\, A$;*
- *cluster variables and coefficients of $\widetilde{\mathcal{A}}$.*

(b) *ψ induces a bijection between*
- *isomorphism classes of reachable basic cluster-tilting objects of $\mathsf{CM}_e\, A$;*
- *clusters of $\widetilde{\mathcal{A}}$.*

Moreover, it commutes with mutation of cluster-tilting objects and mutation of clusters.

To prove Theorem 4.10, we generalize techniques introduced by Jensen-King-Su [JKS] for Grassmannians in type A (see also Subsection 4.1 for Grassmannians of 2-dimensional planes in type A). Meanwhile, we need general results on orders.

We consider an R-order A and an idempotent $e \in A$ such that $B := A/(e)$ is finite dimensional over $\Bbbk$. Let $\mathbb{K} := \Bbbk(\!(t)\!)$ be the fraction field of R, $U := \mathsf{Hom}_A(B, Ae \otimes_R (\mathbb{K}/R))$ and $\mathsf{Sub}\, U$ be the category of B-submodules of objects U^n for $n \geqslant 0$. We consider the exact full subcategory $\operatorname{mod}_e A := \{X \in \operatorname{mod} A \mid eX \in \mathsf{proj}(eAe)\}$ of $\operatorname{mod} A$. Under this setting, we prove the following generalization of a result of [JKS].

Theorem 4.11 ([DI1, Theorem B])**.** *Assume that Ae is injective in $\mathsf{CM}_e\, A$ and has injective dimension at most 1 in $\operatorname{mod}_e A$. Then U is injective in $\operatorname{mod} B$ and there is an equivalence of exact categories*

$$B \otimes_A - : (\mathsf{CM}_e\, A)/[Ae] \xrightarrow{\sim} \mathsf{Sub}\, U.$$

Example 4.12 (from [JKS])**.** Let n be an integer and $k \in \{1, 2, \ldots, n\}$. Let $\widetilde{\Pi} = \Pi(\widetilde{A}_n)$ be the complete preprojective algebra of type $\widetilde{A}_n$. We denote by x (respectively, y) the sum of the arrows of the double quiver of type $\widetilde{A}_n$ going in the decreasing (respectively, increasing) direction. Then $t := xy = yx$ is in the center of $\widetilde{\Pi}$. Let $A := \widetilde{\Pi}/\langle x^k - y^{n+1-k}\rangle$. Denoting $R := \mathbb{C}[\![t]\!]$, we have

$$A \cong \begin{bmatrix} R & R & \cdots & R & t^{-1}R & \cdots & t^{1-k}R \\ tR & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & t^{-1}R \\ t^{k-1}R & & & \ddots & & & R \\ t^kR & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & R \\ t^kR & \cdots & t^kR & t^{k-1}R & \cdots & tR & R \end{bmatrix}_{(n+1)\times(n+1)}.$$

In particular, A is a Gorenstein R-order. Let e be the primitive idempotent at vertex $n+1$. As $eAe \cong R$ is an hereditary order, we have $\mathsf{CM}_e\, A = \mathsf{CM}\, A$. Moreover, $B = A/(e) = \Pi/\langle x^k - y^{n+1-k}\rangle$ where $\Pi = \Pi(A_n)$. As Ae is flat over R, we have a short exact sequence

$0 \to Ae \to Ae \otimes_R \mathbb{K} \to Ae \otimes_R (\mathbb{K}/R) \to 0$, and as B has finite length and $\mathbb{K}$ is injective over R, applying $\mathsf{Hom}_A(B,-)$ to this short exact sequence gives $U \cong \mathsf{Ext}^1_A(B, Ae)$. Then, applying $\mathsf{Hom}_A(-, Ae)$ to the short exact sequence $0 \to AeA \to A \to B \to 0$ gives an exact sequence $0 \to Ae \to \mathsf{Hom}_A(AeA, Ae) \to U \to 0$. Finally, notice that we have

$$\mathsf{Hom}_A(AeA, Ae) = \mathsf{Hom}_A(Ae \otimes_R eA, Ae) \cong \mathsf{Hom}_R(eA, \mathsf{Hom}_A(Ae, Ae)) \cong \mathsf{Hom}_R(eA, R),$$

and, looking at the matrix above, $\mathsf{Hom}_R(eA, R) \cong Af$ where f is the idempotent at vertex $n+1-k$. So $U \cong Bf$. Notice that $\langle x^k - y^{n+1-k} \rangle$ is annihilator of Πf in $\operatorname{mod} \Pi$, so that $\mathsf{Sub}\, U = \mathsf{Sub}(\Pi f)$. So, in this case, Theorem 4.11 gives $(\mathsf{CM}\, A)/[Ae] \cong \mathsf{Sub}(\Pi f)$. Πf is also the injective envelope of the simple Π-module at vertex k. So we recover [JKS, Theorem 4.5].

In Theorem 4.11, direct summands of Ae are projective-injective objects in $\mathsf{CM}_e\, A$ so $\mathsf{Sub}\, U$ is obtained from $\mathsf{CM}_e\, A$ by "removing" some projective-injective objects, hence coefficients from the point of view of the categorification of cluster algebras. Then, the strategy to prove Theorem 4.10 consists in doing a reverse process. Indeed, in view of the explanation at the beginning of this subsection, the category $\mathsf{Sub}\, Q_J$ misses some projective objects.

We also prove a categorical version of Theorem 4.11 in the context of exact categories:

Theorem 4.13 ([DI1, Theorem D])**.** *Let $\mathscr{E}$ be an exact category that is Hom-finite over a field $\Bbbk$. We suppose that*

- *$(\mathscr{A}, \mathscr{B})$ and $(\mathscr{B}, \mathscr{C})$ are torsion pairs in $\mathscr{E}$;*
- *$\mathscr{E}$ has enough projective objects, which belong to $\mathscr{C}$;*
- *There exists a projective object P in $\mathscr{E}$ which is injective in $\mathscr{C}$ and satisfies $\mathscr{A} = \mathsf{add}\, P$;*
- *$\mathscr{B}$ is an abelian category whose exact structure is compatible with that of $\mathscr{E}$.*

Then, there is an equivalence of exact categories

$$\mathscr{C}/[\mathscr{A}] \xrightarrow{\sim} \mathsf{Sub}\, U$$

where U is an (explicitly constructed) injective object of $\mathscr{B}$.

Notice that we need and we give more general versions of Theorems 4.11 and 4.13 in [DI1], with more technical hypotheses and more precise conclusions.

A particular case of Theorem 4.11 occurs if e and g are idempotents of an R-order A such that $Ae \cong \mathsf{Hom}_R(gA, R)$ as left A-modules and $B = A/(e)$ is finite dimensional, then the hypotheses of Theorem 4.11 are satisfied and U is the injective B-module corresponding to the idempotent g. Let us give a motivating example:

Example 4.14. For $n \geqslant 1$, we consider the pair (A, e) defined as follows:

$$A := \begin{bmatrix} R & R \\ (t^n) & R \end{bmatrix} \quad \text{and} \quad e := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

We have $Ae \cong \mathsf{Hom}_R((1-e)A, R)$ and $B = A/(e) \cong \Bbbk[t]/(t^n)$. So according to Theorem 4.11, $(\mathsf{CM}_e\, A)/[Ae] \cong \mathsf{Sub}\, U$. As B is self-injective with one primitive idempotent and U is injective, we have $U = 0$ or $\mathsf{Sub}\, U = \mathsf{Sub}\, B = \operatorname{mod} B$. The first possibility is easy to exclude (we can also compute explicitly $U \cong B$). So we get $(\mathsf{CM}_e\, A)/[Ae] \cong \operatorname{mod} B$. Notice that here $\mathsf{CM}(eAe) = \mathsf{proj}(eAe)$ so $\mathsf{CM}_e\, A = \mathsf{CM}\, A$. We can illustrate this fact by drawing the Auslander-Reiten quivers of $\mathsf{CM}\, A$ and $\operatorname{mod} B$:

$$\begin{tikzcd}
\mathsf{CM}_e\, A : & \begin{bmatrix} R \\ (t^n) \end{bmatrix} \arrow[r, shift left] & \begin{bmatrix} R \\ (t^{n-1}) \end{bmatrix} \arrow[l, shift left, "t"] \arrow[r, shift left] & \begin{bmatrix} R \\ (t^{n-2}) \end{bmatrix} \arrow[l, shift left, "t"] \arrow[r, shift left] & \cdots \arrow[l, shift left, "t"] \arrow[r, shift left] & \begin{bmatrix} R \\ (t) \end{bmatrix} \arrow[l, shift left, "t"] \arrow[r, shift left] & \begin{bmatrix} R \\ R \end{bmatrix} \arrow[l, shift left, "t"] \\
\operatorname{mod} B : \arrow[d, "B \otimes_A -"'] & \Bbbk[t]/(t) \arrow[r, shift left, "t"] & \Bbbk[t]/(t^2) \arrow[l, shift left] \arrow[r, shift left, "t"] & \cdots \arrow[l, shift left] \arrow[r, shift left, "t"] & \Bbbk[t]/(t^{n-1}) \arrow[l, shift left] \arrow[r, shift left, "t"] & \Bbbk[t]/(t^n) \arrow[l, shift left] & \\
& & & & & &
\end{tikzcd}$$

where projective-injective objects are leftmost and rightmost in the first row and only rightmost in the second row. On the other objects, the Auslander-Reiten translation acts as the identity.

Applying Theorem 4.11, we get the following, which is fundamental for Theorem 4.10:

Corollary 4.15 ([DI1, Corollary C]). *Let B be a finite dimensional selfinjective $\Bbbk$-algebra. We define a Gorenstein order A over $R = \Bbbk[\![t]\!]$ and an idempotent e of A by*

$$A := B \otimes_{\Bbbk} \begin{bmatrix} R & R \\ tR & R \end{bmatrix} \quad \text{and} \quad e := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then we have an equivalence of exact categories $(\mathsf{CM}_e\, A)/[Ae] \cong \operatorname{mod} B$ which induces a triangle equivalence $\underline{\mathsf{CM}}_e A \cong \underline{\operatorname{mod}} B$ between stable categories.

Note that in Corollary 4.15, A is generally not an order in the sense of [AG] as $A \otimes_R \Bbbk(\!(t)\!)$ is not usually semisimple. However, we can often replace A by an order in the stronger sense. This question will be investigated further in [DI2].

Corollary 4.15 gives a partial answer to Theorem 4.10, permitting to categorify complete flag varieties using $B = \Pi$. Indeed, we prove in [DI1] that the indecomposable summands of Ae, that we "add" to $\operatorname{mod} \Pi$ as new projective-injective objects are exactly categorifying the missing coefficients of the categorification of Geiss–Leclerc–Schröer.

We finish this subsection by sketching the method used to draw the general conclusion. Roughly speaking, we explain how to "reduce" the size of A in a controlled manner.

Let A be an R-order, e an idempotent of A and B a finite dimensional factor algebra of $A/(e)$. We suppose that there exists an idempotent $g \in A$ such that $Ae \cong \mathsf{Hom}_R(gA, R)$ and $(1-g)\operatorname{soc} B = 0$. Notice that in Corollary 4.15, e and $g = 1 - e$ satisfy these assumption. More generally, if we replace e by a summand e' of it without modifying B, we can still find a corresponding g' (and this is the key point for Theorem 4.10 as we need to be able to reduce the size of e).

We construct a new order A' under this setting. We have a short exact sequence

$$0 \to P \to \widetilde{B} \to B \to 0$$

with $P \in \mathsf{add}\, Ae$ and $\widetilde{B} \in \mathsf{CM}\, A$. Let $W := Ae \oplus \widetilde{B}$ and $A' := \mathsf{End}_A(W)$. We can regard naturally e as an idempotent of A'. We prove that A' is uniquely defined up to Morita equivalence.

Theorem 4.16. *The following assertions hold.*

(a) *We have a canonical isomorphism $B \cong A'/(e)$ of R-algebras.*

(b) *We have an equivalence of exact categories*

$$B \otimes_{A'} - : (\mathsf{CM}_e\, A')/[A'e] \xrightarrow{\sim} \mathsf{Sub}\, Q_g$$

where Q_g is the injective B-module corresponding to the idempotent g.

In the rest of this subsection we give an example illustrating the results of this subsection. Let $B = \Pi$ be the preprojective algebra of type A_3 over the field $\Bbbk$. In other terms

$$\Pi = \Bbbk \left(\begin{tikzcd} 1 \arrow[r, bend left, "\alpha_1"] & 2 \arrow[l, bend left, "\beta_1"] \arrow[r, bend left, "\alpha_2"] & 3 \arrow[l, bend left, "\beta_2"] \end{tikzcd} \right) \Big/ (\alpha_1\beta_1, \alpha_2\beta_2 - \beta_1\alpha_1, \beta_2\alpha_2).$$

This algebra can also be realized as the following subquotient of the matrix algebra $\mathsf{M}_3(\Bbbk[\varepsilon])$:

$$\Pi = \begin{bmatrix} \Bbbk[\varepsilon]/(\varepsilon) & \Bbbk[\varepsilon]/(\varepsilon) & \Bbbk[\varepsilon]/(\varepsilon) \\ (\varepsilon)/(\varepsilon^2) & \Bbbk[\varepsilon]/(\varepsilon^2) & \Bbbk[\varepsilon]/(\varepsilon) \\ (\varepsilon^2)/(\varepsilon^3) & (\varepsilon)/(\varepsilon^2) & \Bbbk[\varepsilon]/(\varepsilon) \end{bmatrix}.$$

Let us denote $R := \Bbbk[\![t]\!]$ and $S := R[\varepsilon]$. The R-order considered in Corollary 4.15 is

$$A := \begin{bmatrix} S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) \\ (\varepsilon)/(\varepsilon^2) & S/(\varepsilon^2) & S/(\varepsilon) & (\varepsilon)/(\varepsilon^2) & S/(\varepsilon^2) & S/(\varepsilon) \\ (\varepsilon^2)/(\varepsilon^3) & (\varepsilon)/(\varepsilon^2) & S/(\varepsilon) & (\varepsilon^2)/(\varepsilon^3) & (\varepsilon)/(\varepsilon^2) & S/(\varepsilon) \\ (t)/(t\varepsilon) & (t)/(t\varepsilon) & (t)/(t\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) \\ (t\varepsilon)/(t\varepsilon^2) & (t)/(t\varepsilon^2) & (t)/(t\varepsilon) & (\varepsilon)/(\varepsilon^2) & S/(\varepsilon^2) & S/(\varepsilon) \\ (t\varepsilon^2)/(t\varepsilon^3) & (t\varepsilon)/(t\varepsilon^2) & (t)/(t\varepsilon) & (\varepsilon^2)/(\varepsilon^3) & (\varepsilon)/(\varepsilon^2) & S/(\varepsilon) \end{bmatrix}.$$

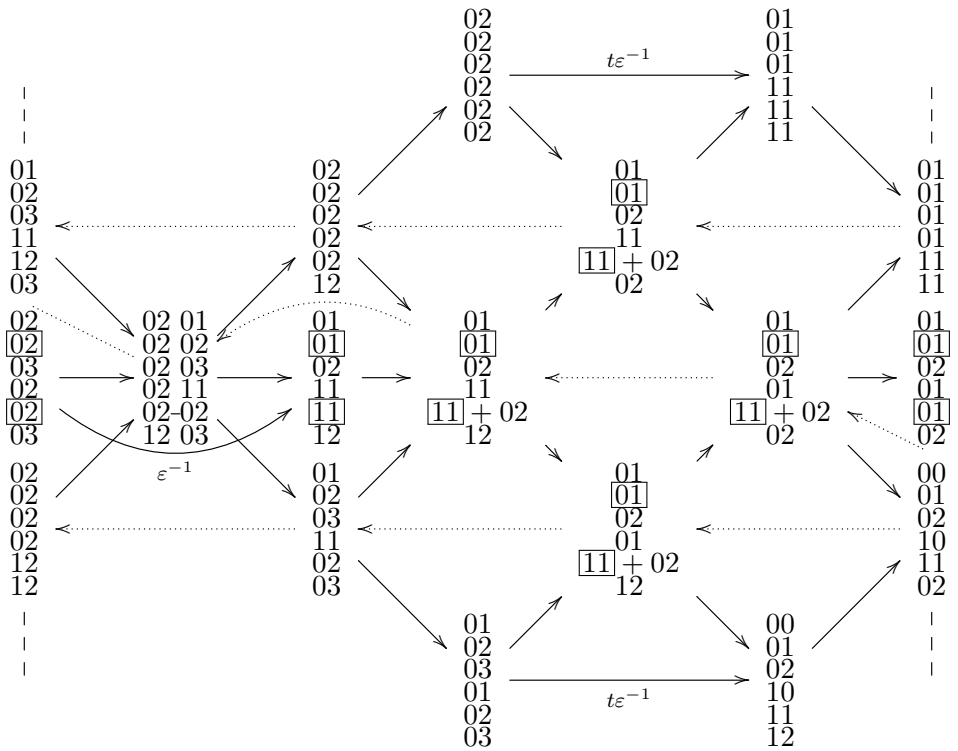


FIGURE 4.17. Auslander-Reiten quiver of $\mathsf{CM}_e\, A$.

On Figure 4.17, we drew the Auslander-Reiten quiver of $\mathsf{CM}_e\, A$, with $ij := (t^i\varepsilon^j)/(t^i\varepsilon^{j+1})$, $\boxed{ij} := (t^i\varepsilon^j)/(t^i\varepsilon^{j+2})$, and ij—$ij := \{(\varepsilon^j p, \varepsilon^j q) \in ij \times ij \mid p - q \in (t,\varepsilon)/(\varepsilon)\}$. All arrows are induced by multiplications by an element of S, which is ± 1 when it is not specified.

Let e_3, e_2, e_1, g_1, g_2 and g_3 be the idempotents corresponding, in this order, to the rows of the matrix. They satisfy $Ae_i \cong \mathsf{Hom}_R(g_iA, R)$ and $Ag_i \cong \mathsf{Hom}_R(e_iA, R)$ as A-modules. We fix the idempotent $e = e_1 + e_2 + e_3$. According to Corollary 4.15, we have an equivalence of exact categories

$$(\mathsf{CM}_e\, A)/[Ae] \cong \operatorname{mod}\Pi.$$

On Figure 4.18, we drew the Auslander-Reiten quiver of $\mathsf{CM}_e\, A$, replacing objects which are not in $\mathsf{add}\, Ae$ by their image by $\Pi \otimes_A -$ in $\mathsf{Sub}\, U = \operatorname{mod}\Pi$ (here $U = \Pi$). We obtain the Auslander-Reiten quiver of $\operatorname{mod}\Pi$ by removing framed objects. The general relation between Auslander-Reiten quivers of $\mathsf{CM}_e\, A$ and $\mathsf{Sub}\, U$ will be discussed in [DI2].

The preimage of ${}_1{}^2{}_3$ by $\Pi \otimes_A -$ is

$$X^\circ := \begin{bmatrix} S/(\varepsilon) & S/(\varepsilon) \\ S/(\varepsilon) & (\varepsilon)/(\varepsilon^2) \\ S/(\varepsilon) & (\varepsilon^2)/(\varepsilon^3) \\ S/(\varepsilon) & (t)/(t\varepsilon) \\ S/(\varepsilon) & \text{—}(\varepsilon)/(\varepsilon^2) \\ (t)/(t\varepsilon) & (\varepsilon^2)/(\varepsilon^3) \end{bmatrix}$$

where $[S/(\varepsilon) \text{ — } (\varepsilon)/(\varepsilon^2)] := \{(x, \varepsilon y) \in S/(\varepsilon) \times (\varepsilon)/(\varepsilon^2) \mid x - y \in (t,\varepsilon)/(\varepsilon)\}$.

Now, we apply Theorem 4.16. Let $e' := e_1 + e_3$ and $B' := \Pi/(\beta_1\alpha_1)$. As a B-module,

$$B' \cong {}^1{}_2{}_3 \oplus {}_1{}^2{}_3 \oplus {}_1{}^2{}^3\,.$$

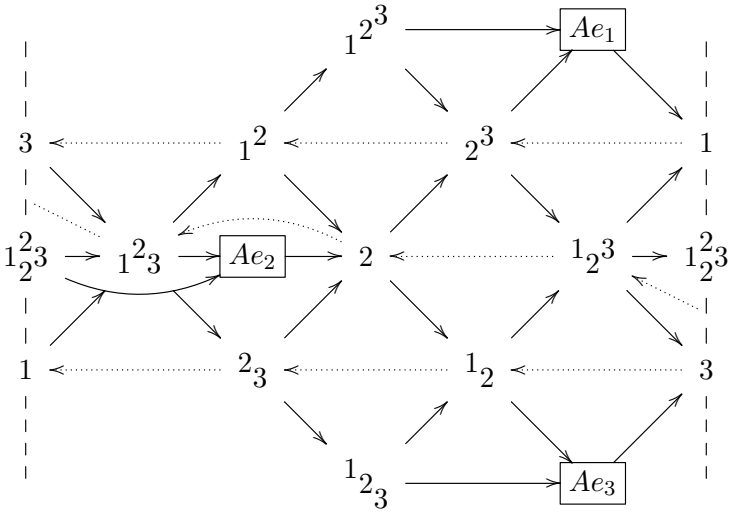


FIGURE 4.18. Auslander-Reiten quiver of $\mathsf{CM}_e\, A$. Objects are represented by their image by $\Pi \otimes_A -$ except objects of $\mathsf{add}\, Ae$.

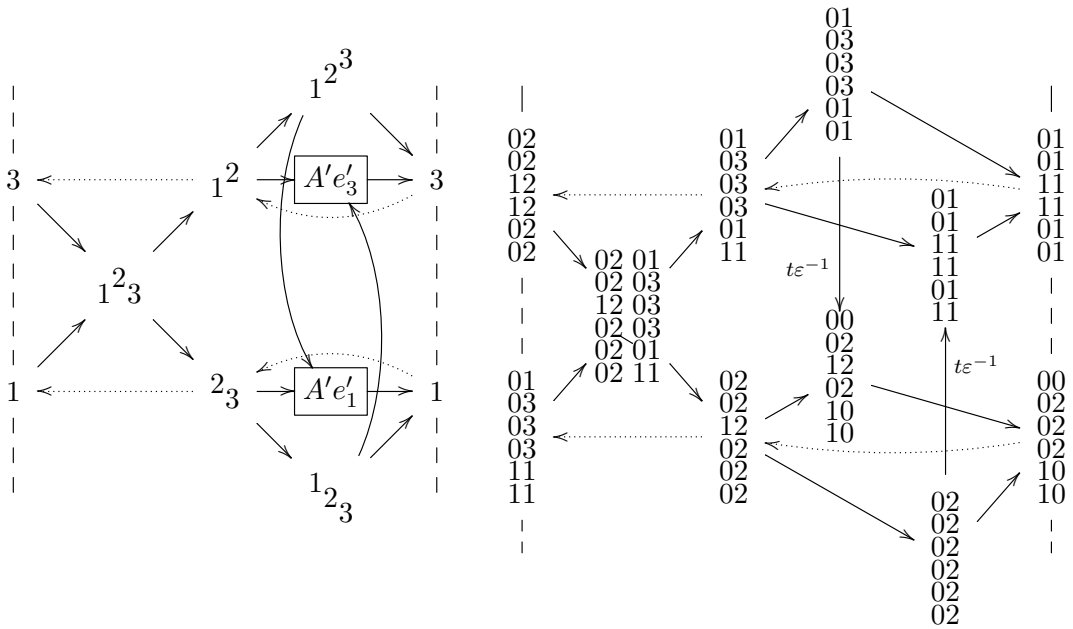


FIGURE 4.19. Auslander-Reiten quiver of $\mathsf{CM}_{e'}\, A'$. On the left diagram, objects are represented by their image by $B' \otimes_{A'} -$ except objects of $\mathsf{add}\, A'e'$.

So we have $W = Ae_1 \oplus Ae_3 \oplus Ag_1 \oplus X^\circ \oplus Ag_3$. Then, $A' := \mathsf{End}_A(W)$ is easy to compute:

$$A' = \begin{bmatrix} S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) \\ (\varepsilon^2)/(\varepsilon^3) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & (\varepsilon^2)/(\varepsilon^3) & (\varepsilon^2)/(\varepsilon^3) \\ (t\varepsilon^2)/(t\varepsilon^3) & (t)/(t\varepsilon) & S/(\varepsilon) & (t)/(t\varepsilon) & (\varepsilon^2)/(\varepsilon^3) & (\varepsilon^2)/(\varepsilon^3) \\ (\varepsilon^2)/(\varepsilon^3) & (t)/(t\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & (\varepsilon^2)/(\varepsilon^3) & (\varepsilon^2)/(\varepsilon^3) \\ (t)/(t\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & S/(\varepsilon) \\ (t)/(t\varepsilon) & (t)/(t\varepsilon) & S/(\varepsilon) & S/(\varepsilon) & (t)/(t\varepsilon) & S/(\varepsilon) \end{bmatrix}$$

where $[S/(\varepsilon) — S/(\varepsilon)] := \{(x, y) \in S/(\varepsilon) \times S/(\varepsilon) \mid x - t \in (t)/(t\varepsilon)\}$.

Thanks to Theorem 4.16, we have $(\mathsf{CM}_{e'} A')/[Ae']$ is equivalent to the subcategory of $\operatorname{mod} \Pi$ consisting of modules whose socle is supported at vertices 1 and 3. To illustrate this fact, we give two representations of the Auslander-Reiten quiver of $\mathsf{CM}_{e'} A'$ on Figure 4.19.

4.3. **Open problems.** A natural interesting question about the cluster character ψ of Theorem 4.10 is the following one:

Question 4.20. *Is there a direct, natural formula for ψ without using the forgetful functor $\mathsf{CM}_e A \to \mathsf{Sub}\, Q_J$?*

Indeed, in [DI1], we use the cluster character defined on $\mathsf{Sub}\, Q_J$ in [GLS2] and we extend it combinatorially to $\mathsf{CM}_e A$, defining explicitly ψ on direct summands of Ae.

We obtain interesting examples of orders where $\Lambda/(e)$ are quotient of preprojective algebras. On the other hand, these examples can sometimes be interpreted as coming from isolated singularities (see also [JKS]). A typical way to construct such an example is to consider a curve $\mathscr{S}$ of $\mathbb{A}^2(\mathbb{C})$ and a subgroup G of $\mathsf{SL}_2(\mathbb{C})$ stabilizing $\mathscr{S}$. Then, the skew-group algebra $\mathbb{C}[\mathscr{S}] * G$ is an order which, under good geometrical hypotheses, satisfy the assumption of our work [DI1]. There are still numerous families of geometrical situations to explore and which should permit to construct new families of orders which are still not studied in the literature.

5. Relation between cluster-tilting theory and τ-tilting theory

This very short section is devoted to relate two main objects of this document. Namely, cluster-tilting objects and τ-tilting objects.

Let $\mathscr{C}$ be a $\Bbbk$-linear, Hom-finite, Krull-Schmidt, 2-Calabi-Yau triangulated category with a basic cluster-tilting object $T = T_1 \oplus \cdots \oplus T_n$ as at the beginning of Section 4. We remind the reader that we have $\mathscr{C} = \mathsf{add}\, T * \mathsf{add}\, T[1]$, *i.e.* all objects X of $\mathscr{C}$ appear in a triangle $T^0 \to X \to T^1[1]$ with $T^0, T^1 \in \mathsf{add}\, T$, see for example [KR, §2.1]. We denote the set of isomorphism classes of basic cluster-tilting objects in $\mathscr{C}$ by $\mathsf{c\text{-}tilt}\,\mathscr{C}$.

We note that in general the indecomposable direct summands of T do not form a basis of the Grothendieck group of $\mathscr{C}$. Thus, we consider the split Grothendieck group $K_0^{\oplus}(T)$ of the additive category $\mathsf{add}\, T$. That is, $K_0^{\oplus}(T)$ is the quotient of the free abelian group generated by the isomorphism classes of objects in $\mathsf{add}\, T$ by the subgroup generated by all elements of the form

$$[T' \oplus T''] - [T'] - [T''].$$

Thus we have $K_0^{\oplus}(T) = \mathbb{Z}[T_1] \oplus \mathbb{Z}[T_2] \oplus \cdots \oplus \mathbb{Z}[T_n]$.

Let $M \in \mathscr{C}$. There exists a triangle

$$T^1 \to T^0 \xrightarrow{f} M \to T^1[1]$$

such that $T^0, T^1 \in \mathsf{add}\, T$ and f is a minimal right $\mathsf{add}\, T$-approximation. The *index of M with respect to T* is defined by

$$\mathsf{ind}_T(M) := [T^0] - [T^1] \in K_0^{\oplus}(T).$$

Dually, consider a triangle $M \xrightarrow{g} ({}^0T)[2] \to ({}^1T)[2] \to M[1]$ with ${}^0T, {}^1T \in \mathsf{add}\, T$ and g is a minimal left $\mathsf{add}\, T[2]$-approximation. The *coindex of M with respect to T* is defined by

$$\mathsf{coind}^T(M) := [{}^0T] - [{}^1T] \in K_0^{\oplus}(T).$$

Note that we have $\mathsf{coind}^T(M) = -\,\mathsf{ind}^T(M[-1])$

Let $M = M_1 \oplus \cdots \oplus M_n \in \mathsf{c\text{-}tilt}\,\mathscr{C}$. The G*-matrix of M with respect to T* is the integer matrix $\mathsf{G}(T, M) := [g_T^{M_1} \mid \cdots \mid g_T^{M_n}]$ where $g_T^{M_i}$ is the column vector corresponding to $\mathsf{ind}_T(M_i)$ in the ordered basis $\{[T_1], \ldots, [T_n]\}$. Similarly, we define the C*-matrix of M with respect to T* using coindices, and denote it by $\mathsf{C}(T, M)$.

Let $A := \mathsf{End}_{\mathscr{C}}(T)$. Recall from [KR, Proposition 2(c)] that the functor $\mathsf{Hom}_{\mathscr{C}}(T, -) \colon \mathscr{C} \to \operatorname{mod} A$ induces an equivalence of categories

$$F \colon \mathscr{C}/[T[1]] \xrightarrow{\sim} \operatorname{mod} A$$

where $[T[1]]$ is the ideal of $\mathscr{C}$ of morphisms that factor through $\mathsf{add}\, T[1]$. Moreover, it is shown in [AIR, Theorem 4.1] that F induces a bijection

$$\mathsf{c\text{-}tilt}\,\mathscr{C} \xrightarrow{F} \tau\text{-}\mathsf{tilt\text{-}pair}\, A$$

given by $M = (X \oplus T'[1]) \mapsto (FX, FT')$ where $T'[1]$ satisfies $\mathsf{add}\, M \cap \mathsf{add}\, T[1] = \mathsf{add}\, T'[1]$ and X has no indecomposable direct summands in $\mathsf{add}\, T[1]$. Notice that, under this bijection, the exchange graph of cluster-tilting objects of $\mathscr{C}$ is isomorphic to the underlying non-oriented graph of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt\text{-}pair}\, A)$. In other terms, for each cluster-tilting object $T \in \mathscr{C}$, there is a canonical lattice structure over $\mathsf{c\text{-}tilt}\,\mathscr{C}$ such that T is the biggest element.

In [DIJ], we obtain a new proof of the following result in cluster-tilting theory, see for example [Nak, Theorem 4.1 and Remark 4.2]. This new proof uses the expression of G-matrices and C-matrices in $\tau\text{-}\mathsf{tilt\text{-}pair}\, A$ via the bijection F.

Theorem 5.1. *If M is a cluster-tilting object in $\mathscr{C}$, then*

$$\mathsf{C}(M[-2], T) = \mathsf{G}(T, M)^{-1}.$$

6. Algebras of partial triangulations

The aim of this section is to present algebras of partial triangulations of marked surfaces. This class of algebras, which always have finite rank, contains classical Jacobian algebras of triangulations of marked surfaces and Brauer graph algebras. We discuss representation theoretical properties and derived equivalences. All results are proven in [Dem4], under slightly milder hypotheses.

There is also a part of [Dem4] dealing with *frozen algebras of partial triangulations*. They generalize frozen Jacobian algebras as defined in Subsection 4.1. However, results obtained about these algebras seem at the moment less important than the ones concerning algebras of partial triangulations.

6.1. **The algebra of a partial triangulation.** Let $\Bbbk$ be a unital ring and Σ be a connected compact oriented surface with or without boundary. We fix a non-empty finite set $\mathbb{M} \subset \Sigma$ of marked points (some of them may be on the boundary $\partial\Sigma$). For each $M \in \mathbb{M}$, we fix an invertible coefficient $\lambda_M \in \Bbbk^\times$ and a multiplicity $m_M \in \mathbb{Z}_{>0}$. For simplicity, we suppose here that if Σ is a sphere then $\#\mathbb{M} \geq 5$ and if Σ is a disc then $\#\mathbb{M} \geq 3$.

Definition 6.1. An *arc* on $(\Sigma, \mathbb{M})$ is a continuous map $u : [0,1] \to \Sigma$ satisfying:

- The restriction of u to $(0,1)$ is an embedding into $\Sigma \setminus \mathbb{M}$;
- Extremities 0 and 1 are mapped to $\mathbb{M}$.

We consider arcs up to homotopy relative to their endpoints in $\Sigma \setminus \mathbb{M}$. Moreover, we exclude arcs that are homotopic to a marked point or to a boundary component, that is the closure of a connected component of $\partial\Sigma \setminus \mathbb{M}$.

In this section, for simplicity, we exclude arcs that are loops enclosing a unique marked point $M \in \mathbb{M}$ with $m_M \leqslant 2$. We do not make this restriction in [Dem4].

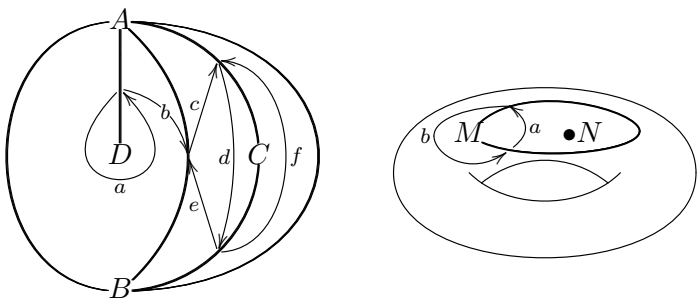
Definition 6.2. We say that two arcs u and v are *compatible* if, up to homotopy, they are non-crossing. Then, a *partial triangulation of* $(\Sigma, \mathbb{M})$ is a set σ of arcs of $(\Sigma, \mathbb{M})$ that are pairwise compatible. If σ is a maximal partial triangulation and each connected component of $\partial\Sigma$ contains at least a marked point, σ is called a *triangulation*.

In order to define the algebra $\Delta_\sigma = \Delta_\sigma^\lambda$ of a partial triangulation σ, we first need to construct a quiver Q_σ.

Definition 6.3. The quiver Q_σ has set of vertices σ, and arrows are winding in Σ between successive arcs around marked points counter-clockwisely. We call *bouncing path* of Q_σ any path of length 2 consisting of two arrows that are not successive around the same endpoint.

We give two examples to illustrate this definition:

Example 6.4. In the left example Σ is a disc with four marked points A, B, C and D (A and B are on the boundary). In the right one, Σ is a torus with two marked points M and N. The partial triangulations are depicted with thick lines.



Bouncing paths of the first example are ab, cd, de and ec. Bouncing paths of the second example are a^2 and b^2.

To define the algebra Δ_σ, we need two more combinatorial concepts:

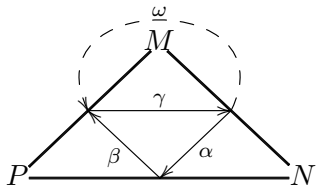
Definition 6.5. For each arc $u \in \sigma$ and endpoint M of u, we denote by $\omega_{u,M}$ the path of Q_σ going from u to u winding once around M if $M \notin \partial\Sigma$ and $\omega_{u,M} = 0$ if $M \in \partial\Sigma$.

Definition 6.6. A *small triangle* of σ is a triple (u, v, w) of arcs of σ together with three marked points M, N and P such that

- u is incident to M and N;
- v is incident to N and P;
- w is incident to P and M;
- The union of u, v and w encloses clockwisely (in the order u, v, w) a disc without marked point inside.

We now define the *algebra of the partial triangulation* σ by $\Delta_\sigma = \Delta_\sigma^\lambda := \Bbbk Q_\sigma / I_\sigma$ where I_σ is the ideal generated by the following relations:

(a) For each $u \in \sigma$ that joins $M \in \mathbb{M}$ to $N \in \mathbb{M}$, we require $\lambda_M \omega_{u,M}^{m_M} = \lambda_N \omega_{u,N}^{m_N}$.
(b) For each $u \in \sigma$ with endpoint M, we require $\omega_{u,M}^{m_M} q = 0$ for any arrow q of Q_σ.
(c) Suppose that σ contains a small triangle M, N, P as in the following picture:



where α, β and γ are arrows, ω is a the only possible simple path winding around M if $M \notin \partial\Sigma$ and $\omega = 0$ if $M \in \partial\Sigma$. We require the relation $\alpha\beta = \lambda_M (\omega\gamma)^{m_M - 1}\omega$.
(d) For any bouncing path $\alpha\beta$ that does not appear in case (c), we require $\alpha\beta = 0$.

Example 6.7. Relations for the left partial triangulation of Example 6.4 are:

$$a^{m_D} = (df)^{m_C} = (fd)^{m_C} = 0 \qquad \text{(a)}$$
$$ab = 0 \qquad \text{(d)}$$
$$cd = de = 0 \qquad \text{(c)}$$
$$ec = \lambda_C (fd)^{m_C - 1} f \qquad \text{(c)}$$

and other relations are redundant.

Relations for the right partial triangulation are:

$$(ab)^{m_M} = (ba)^{m_M} \qquad \text{(a)}$$
$$a^2 = b^2 = 0. \qquad \text{(d)}$$

It turns out that algebras of partial triangulations are particularly well-behaved. A first result about them is that this definition is compatible with the naive notion of a sub-partial triangulation:

Theorem 6.8. *Let $\tau \subset \sigma$. Then we have*

$$\Delta_\tau \cong e_\tau \Delta_\sigma e_\tau$$

where e_τ is the sum of the primitive idempotents of Δ_σ corresponding to the arcs of τ.

Note that we have naturally $\Bbbk Q_\tau \subset \Bbbk Q_\sigma$. However, relations as defined in this note do not go through this inclusion. We have to take a more complicated variant of these relations, giving an isomorphic algebra, to obtain Theorem 6.8.

6.2. **Brauer graph algebras and Jacobian algebras of surfaces.** We explain here that the class of algebras of partial triangulations contains two important classes of algebras.

Theorem 6.9. *If σ contains no small triangle, neither arc incident to $\partial\Sigma$, then Δ_σ is the* Brauer graph algebra *of σ considered as a* ribbon graph. *Moreover, any Brauer graph algebra is the algebra of a partial triangulation of a surface without boundary.*

We will not recall what a Brauer graph algebra is. For more details, see for example [Rog] or [WW]. However, this definition is very close to the definition of the algebra of a partial triangulation and Theorem 6.9 is mostly straightforward.

Theorem 6.10. *If all m_M are invertible in $\Bbbk$ and σ is a triangulation, then Δ_σ is the Jacobian algebra of a quiver with potential (Q_σ, W_σ). To define W_σ, consider the set E of small triangles T of σ up to rotation and for each of them denote by α_T, β_T and γ_T the three arrows as in the figure defining relation (c) earlier. Then, for each $M \in \mathbb{M}$ take arbitrarily an arc u_M incident to M. Then*

$$W_\sigma := \sum_{T\in E} \alpha_T\beta_T\gamma_T - \sum_{M\in\mathbb{M}} \frac{\lambda_M}{m_M}\omega_{u,M}^{m_M}$$

(as usual for potentials, terms are only well defined up to cyclic permutations).

Notice that if $m_M = 1$ for all $M \in \mathbb{M}$, we recover the usual Jacobian algebra of a surface as defined in [LF]. Recall that the Jacobian algebra of (Q_σ, W_σ) is the quotient of the completed path algebra $\widehat{\Bbbk Q_\sigma}$ by the cyclic derivatives of W_σ. So we get here an improvement as Δ_σ is directly defined from $\Bbbk Q_\sigma$ without completion.

6.3. **Algebraic properties of Δ_σ.** We have the following result about Δ_σ:

Theorem 6.11. *The $\Bbbk$-algebra Δ_σ is a free $\Bbbk$-module of rank*

$$\sum_{M\in\mathbb{M}\cap\partial\Sigma} \frac{d_M(d_M-1)}{2} + \sum_{M\in\mathbb{M}\cap(\Sigma\setminus\partial\Sigma)} m_M d_M^2 + f$$

where, for $M \in \mathbb{M}$, d_M is the degree of M in the graph σ (without counting boundary components), and f is the number of arcs in σ with both endpoints on boundaries.

Example 6.12. The algebra of the left partial triangulation of Example 6.4 has rank $5+4m_C+m_D$, and the right one has rank $4m_M$.

More precisely, there is a $\Bbbk$-basis of Δ_σ consisting of all strict and non-idempotent prefixes of all $\omega_{u,M}^{m_M}$, together with primitive idempotents and elements $\lambda_M\omega_{u,M}^{m_M} = \lambda_N\omega_{u,N}^{m_N}$.

The following property generalizes a known result for Brauer graph algebras and Jacobian algebras of surfaces without boundary:

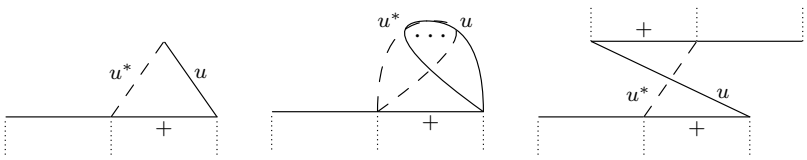
Theorem 6.13. *If σ has no arc incident to the boundary, then Δ_σ is a symmetric $\Bbbk$-algebra (*i.e. $\mathsf{Hom}_\Bbbk(\Delta_\sigma, \Bbbk) \cong \Delta_\sigma$ *as Δ_σ-bimodules).*

6.4. **Representation type of Δ_σ.** The next theorem permits to expect that the classification of Δ_σ-modules is possible:

Theorem 6.14. *If $\Bbbk$ is an algebraically closed field, then Δ_σ is of tame representation type.*

The proof of this result relies on a deformation theorem by Crawley-Boevey [CB]. Indeed, the relations defining Δ_σ can be deformed to the relations of a Brauer graph algebra in a suited manner. Moreover, Brauer graph algebras are of tame representation type. Notice that unfortunately, these techniques do not permit to deduce directly the classification of Δ_σ-modules even though modules over Brauer graph algebras are known.

6.5. **Flip of partial triangulations and derived equivalences.** Finally, we give a flip leading to derived equivalences. For an arc u in σ such that arcs marked by $+$ in the following diagrams are also in σ (in particular, they are not in $\partial\Sigma$), we define $\mu_u(\sigma)$ by replacing u by u^* defined in the following way:



(F1) (F2) (F3)

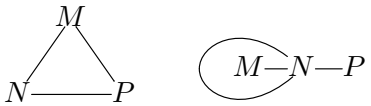
We also define coefficients $\mu_u(\lambda)$. For a marked point $M \in \mathbb{M}$, $\mu_u(\lambda)_M = \lambda_M$ except in the following cases:

- In case (F1), if M is the topmost vertex of the figure then $\mu_u(\lambda)_M = -\lambda_M$.
- In case (F1), if M is the rightmost vertex of the figure then $\mu_u(\lambda)_M = (-1)^{m_M}\lambda_M$.
- In case (F2), if M is the unique marked point enclosed by u then $\mu_u(\lambda)_M = -\lambda_M$.

Then we get the following result:

Theorem 6.15. *There is a derived equivalence between Δ_σ^λ and $\Delta_{\mu_u(\sigma)}^{\mu_u(\lambda)}$.*

Example 6.16. We consider the two following partial triangulations of a disc with three marked points, none of them are in $\partial\Sigma$:



They are related by a flip so the following algebras, obtained for $\lambda_M = \lambda_N = \lambda_P = 1$ and $m_M = m_N = m_P = m$ are derived equivalent:

$$\frac{\Bbbk\left(\begin{tikzcd} & \bullet \arrow[dl, bend right, "y"'] \arrow[dr, bend left, "y"] & \\ \bullet \arrow[ur, bend right, "x"'] \arrow[rr, bend right, "y"'] & & \bullet \arrow[ul, bend left, "x"] \arrow[ll, bend left, "x"] \end{tikzcd}\right)}{(x^2 - (yx)^{m-1}y, y^2)} \quad \text{and} \quad \frac{\Bbbk\left(\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] & \bullet \arrow[l, bend right, "\beta_2"'] \arrow[r, bend right, "\beta_1"'] & \bullet \arrow[l, bend right, "\gamma_2"'] \arrow[r, bend right, "\gamma_1"'] & \bullet \arrow[loop right, "\delta"] \end{tikzcd}\right)}{\left(\begin{array}{c} \beta_2\alpha - (\gamma_1\gamma_2\beta_2\beta_1)^{m-1}\gamma_1\gamma_2\beta_2, \\ \alpha\beta_1 - (\beta_1\gamma_1\gamma_2\beta_2)^{m-1}\beta_1\gamma_1\gamma_2, \\ \beta_1\beta_2 - \alpha^{m-1}, \gamma_1\delta, \delta\gamma_2, \gamma_2\gamma_1, \\ \delta^m - (\gamma_2\beta_2\beta_1\gamma_1)^m \end{array}\right)}.$$

6.6. **Open problems.** As algebras of partial triangulations are of tame representation type, it is theoretically possible to classify their indecomposable representations, or at least to understand the structure of their Auslander-Reiten quiver (connected components especially). A natural question is, for derived equivalences of Subsection 6.5, to understand their restrictions to module categories (it is expected that these derived equivalences induce bijections between important classes of modules, as tilting complexes share most of their summands with the algebra itself). The following question should be easy to answer:

Question 6.17. *For a partial triangulation σ and its tilt $\mu_u(\sigma)$ at an edge u, do we have an equivalence*

$$\frac{\operatorname{mod}\Delta_\sigma^\lambda}{[S_u]} \cong \frac{\operatorname{mod}\Delta_{\mu_u(\sigma)}^{\mu_u(\lambda)}}{[S_{u^*}]}$$

where S_u and S_{u^*} are simple modules at edges u and u^*?

This would be a natural extension of some results that are known for mutations of Jacobian algebras.

We also have the project with Aaron Chan to classify τ-rigid modules over Δ_σ. They should correspond to most *laminations* as studied in [FT] and **g**-vectors would correspond to so-called *shear coordinates*.

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Combinatorics of Mutations in Representation Theory

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Résumé. Ce mémoire d'habilitation à diriger des recherches prend la forme d'une introduction détaillée à mon domaine de recherche et à mes recherches. Je discute certains problèmes à l'interface de la théorie des représentations d'algèbres de dimension finie et de la combinatoire. Une première partie introduit les représentations d'algèbres de dimension finie et les modules de Cohen-Macaulay, se focalisant sur la théorie d'Auslander-Reiten. Ces techniques sont ensuite utilisées pour développer certains thèmes de mes recherches. Premièrement, une technique pour comprendre la structure de treillis des classes de torsion sur une algèbre de dimension finie, en particulier leur quotients. Deuxièmement, plusieurs catégorifications d'algèbres amassées utilisant des catégories de modules de Cohen-Macaulay : en utilisant des triangulations de polygones d'une part, et certains résultats plus abstraits permettant en particulier de catégorifier les algèbres de coordonnées multihomogènes sur les variétés de drapeaux partiels. Enfin, une famille d'algèbres de nature combinatoire à partir de triangulations partielles de surfaces, et certaines de leur propriétés les plus notables : l'existence d'une mutation donnant lieu à des équivalences dérivées et le fait que ces algèbres sont apprivoisées (c'est-à-dire que leurs catégories de modules sont théoriquement plus faciles à comprendre).

Abstract. This memoir is a survey about my research topics and my own results. I discuss certain problems situated at the boundary between representation theory of finite dimensional algebras and combinatorics. The first part is an introduction about representations of finite dimensional algebras and Cohen-Macaulay modules, with a focus on Auslander-Reiten theory. These techniques are then exploited to develop certain topics of my research. First, a technique to understand the lattice structure of the set of torsion classes on a finite dimensional algebra, in particular the lattice quotients of these lattices. Secondly, several categorifications of cluster algebras via Cohen-Macaulay modules: using triangulations of polygons on the one hand, and more abstract results which permit in particular to categorify all multihomogeneous coordinate rings on partial flag varieties, on the other hand. Lastly, a family of algebras constructed combinatorially from partial triangulations of surfaces and certain of their most noticeable properties: the existence of a mutation giving rise to derived equivalences, and their tameness (in other terms, the fact that the module categories of these algebras can theoretically be understood more easily).

Mots clés. Représentations d'algèbres de dimension finie, Modules de Cohen-Macaulay, Catégorifications d'algèbres amassées, Théorie d'Auslander-Reiten, Classes de torsion, Treillis.

Keywords. Representation theory of finite dimensional algebras, Cohen-Macaulay modules, Categorification of cluster algebras, Auslander-Reiten theory, Torsion classes, Lattices.