Geometric methods in representation theory
Assessment 2

To hand in July 7th

Problem 1: Let A be a finite-dimensional algebra. Recall that the projective dimension $\operatorname{prdim} X$ of $X \in \operatorname{mod} kQ$ is the length of a (minimal) projective resolution of X (hence X is projective if and only if $\operatorname{prdim} X = 0$). We consider a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod} kQ$.

(1) Prove that for any projective resolutions $P_X^\bullet$ of X and $P_Z^\bullet$ of Z then there exists a projective resolution $P_Y^\bullet$ of Y such that for any i, $P_Y^i = P_X^i \oplus P_Z^i$ [This is called the horseshoe lemma].

(2) Deduce that $\operatorname{prdim} Y \leq \max(\operatorname{prdim} X, \operatorname{prdim} Z)$.

In Problems 2 and 3, we consider an acyclic quiver Q. Thus, the path algebra kQ is finite-dimensional.

Problem 2:

(1) Let $i \in Q_0$. Compute a minimal resolution of the simple module S_i supported at i.

(2) Deduce that any $M \in \operatorname{mod} kQ$ has projective dimension at most 1.

(3) Prove that any submodule of a projective kQ-module is again projective.

(4) For any $i, j \in Q_0$ and $n \in \{0, 1, 2, \dots\}$, compute $\dim \operatorname{Ext}^n_{kQ}(S_i, S_j)$ in terms of the quiver Q (Recall that $\operatorname{Ext}^0_{kQ} = \operatorname{Hom}^0_{kQ}$).

Problem 3: For M and N in $\operatorname{mod} kQ$, we consider the Euler form

$$\langle M, N \rangle := \dim \operatorname{Hom}_{kQ}(M, N) - \dim \operatorname{Ext}^1_{kQ}(M, N).$$

(1) Let $\mathbf{d} := \mathbf{dim}\, M$ and $\mathbf{e} := \mathbf{dim}\, N$. Prove that $\langle M, N \rangle = \sum_{i \in Q_0} d_i e_i - \sum_{q \in Q_1} d_{s(q)} e_{t(q)}$. (*Hint:* do it for simple modules and by induction on the dimension.)

(2) Justify that for $M \in \operatorname{mod} kQ$ and $\mathbf{d} := \mathbf{dim}\, M$, we have $\dim \operatorname{GL}(\mathbf{d}) = \dim \mathcal{O}_M + \dim \operatorname{End}_{kQ} M$ where $\mathcal{O}_M$ is the orbit of M in $\mathcal{M}\mathrm{od}(kQ, \mathbf{d})$.

(3) Compute $\dim \mathcal{M}\mathrm{od}(kQ, \mathbf{d})$ and $\dim \operatorname{GL}(\mathbf{d})$.

(4) Deduce that $\dim \operatorname{Ext}^1_{kQ}(M) = \operatorname{codim} \mathcal{O}_M (:= \dim \mathcal{M}\mathrm{od}(kQ, \mathbf{d}) - \dim \mathcal{O}_M)$.