Geometric methods in representation theory
Assessment 1

To hand in June 9th

Problem 1 : Consider the quiver

$$Q = 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and the ideal $I = (\alpha\beta, \beta\alpha)$ of the path algebra $\mathbb{C}Q$. We denote $\Lambda = \mathbb{C}Q/I$ and fix the dimension vector $\mathbf{d} = (1, 2)$.

(i) Compute the representation variety $\mathcal{M}\text{od}(\mathbb{C}Q, \mathbf{d})$.
(ii) Compute $\mathcal{M}\text{od}(\Lambda, \mathbf{d})$.
(iii) Classify orbits of the action of $\mathrm{GL}(\mathbf{d})$ over $\mathcal{M}\text{od}(\mathbb{C}Q/I, \mathbf{d})$.
(iv) For each orbit $\mathcal{O}$, give an $x \in \mathcal{O}$ and compute the corresponding Λ-module $\mathbb{C}_x$.
(v) For each orbit $\mathcal{O}$, compute $\dim \mathcal{O}$.
(vi) Compute $\dim \mathcal{M}\text{od}(\Lambda, \mathbf{d})$.

Problem 2 : We consider a quiver Q and an ideal $I \subset \mathbb{C}Q$. Let $\Lambda = \mathbb{C}Q/I$. Let $\mathbf{d}$ be a dimension vector for Q. Prove that for $x \in \mathcal{M}\text{od}(\Lambda, \mathbf{d})$, we have $\mathrm{Stab}_{\mathrm{GL}(\mathbf{d})}(x) \cong \mathrm{Aut}_\Lambda(\mathbb{C}_x)$ where $\mathrm{Stab}_{\mathrm{GL}(\mathbf{d})}(x) = \{g \in \mathrm{GL}(\mathbf{d}) \mid gxg^{-1} = x\}$ and $\mathrm{Aut}_\Lambda(\mathbb{C}_x)$ is the automorphism group of the representation $\mathbb{C}_x$ corresponding to x.

Problem 3 : The aim is to prove the following theorem that has not been proven in the course:

Theorem. *If $\pi : \mathcal{X} \to \mathcal{Y}$ is an algebraic map, then $x \mapsto \dim_x \pi^{-1}(\pi(x))$ is upper semicontinuous.*

We denote $Z(\pi, n) = \{x \in \mathcal{X}, \dim_x \pi^{-1}(\pi(x)) \geq n\}$.

(i) Let $\mathcal{X} = \bigcup_{i=1}^{\ell} \mathcal{X}_i$ be the decomposition if $\mathcal{X}$ into irreducible components. Prove that

$$Z(\pi, n) = \bigcup_{i=1}^{\ell} Z(\pi|_{\mathcal{X}_i}, n).$$

Suppose in Questions (ii), (iii) and (iv) that $\mathcal{X}$ is indecomposable and $\mathcal{Y} = \pi(\mathcal{X})$.

(ii) Using the Key Lemma of Chapter IV, prove that there exists an open subset U of $\mathcal{Y}$ such that, for all $y \in U$, $\dim \pi^{-1}(\{y\}) = \dim \mathcal{X} - \dim \mathcal{Y}$.
(iii) Deduce that either $Z(\pi, n) = \mathcal{X}$ or $Z(\pi, n) = Z(\pi|_{\mathcal{X} \setminus \pi^{-1}(U)}, n)$.
(iv) Conclude that, in this case, $Z(\pi, n)$ is closed in $\mathcal{X}$ (you can do an induction on $\dim \mathcal{X}$).

We go back to the general case where $\mathcal{X}$ is not necessarily indecomposable and $\mathcal{Y}$ is not necessarily $\pi(\mathcal{X})$.

(v) Prove the theorem.

Problem 4 : We consider the algebra Λ of Problem 1. We denote by S_1 the simple module at vertex 1, that is the only Λ-module with dimension vector $(1, 0)$. In the same way, S_2 is the simple module at vertex 2.

(i) Compute all isomorphism classes of indecomposable projective Λ-modules.
(ii) Compute a projective resolution of S_1 and a projective resolution of S_2.
(iii) For $i \in \mathbb{N}$, compute $\mathrm{Ext}^i_\Lambda(S_1, S_1)$, $\mathrm{Ext}^i_\Lambda(S_1, S_2)$, $\mathrm{Ext}^i_\Lambda(S_2, S_1)$ and $\mathrm{Ext}^i_\Lambda(S_2, S_2)$.
(iv) Compute bases of $\mathrm{Ext}^1_\Lambda(S_1, S_1)$, $\mathrm{Ext}^1_\Lambda(S_1, S_2)$, $\mathrm{Ext}^1_\Lambda(S_2, S_1)$ and $\mathrm{Ext}^1_\Lambda(S_2, S_2)$ in terms of short exact sequences.