Complex Analysis - Mid-term exam - Answers

Exercise 1 (10%) : Compute $\operatorname{Arg}\left(-5+5\sqrt{3}i\right)$.

Answer:

We have:

$$|z| = \sqrt{5^2+(5\sqrt{3})^2} = \sqrt{25+25\times 3} = \sqrt{100} = 10$$

so

$$\frac{z}{|z|} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = e^{\frac{2\pi}{3}i}$$

and finally

$$\operatorname{Arg}\left(-5+5\sqrt{3}i\right) = \frac{2\pi}{3}.$$

Exercise 2 (10%) : See other sheet.

Exercise 3 (20%) : Solve the equations or system of equations (in $\mathbb{C}$):

1. $x^3 = 2i$.

   Answer:

   We wants to compute the cube roots of $2i$. It is easy to compute the polar form of $2i$. Indeed, $|2i| = 2$ and $\operatorname{Arg}(i) = \pi/2$:

   $$2i = 2e^{\frac{\pi}{2}i}.$$

   We know that a nonzero complex number admits exactly three cube roots. These cube roots are

   $$z_0 = \sqrt[3]{2}e^{\frac{\frac{\pi}{2}}{3}i} = \sqrt[3]{2}e^{\frac{\pi}{6}i} = \sqrt[3]{2}\cos\frac{\pi}{6} + \sqrt[3]{2}\left(\sin\frac{\pi}{6}\right)i = \frac{\sqrt[3]{2}\times\sqrt{3}}{2} + \frac{\sqrt[3]{2}}{2}i;$$

   $$z_1 = \sqrt[3]{2}e^{\frac{\frac{\pi}{2}+2\pi}{3}i} = \sqrt[3]{2}e^{\frac{5\pi}{6}i} = \sqrt[3]{2}\cos\frac{5\pi}{6} + \sqrt[3]{2}\left(\sin\frac{5\pi}{6}\right)i = -\frac{\sqrt[3]{2}\times\sqrt{3}}{2} + \frac{\sqrt[3]{2}}{2}i;$$

   $$z_2 = \sqrt[3]{2}e^{\frac{\frac{\pi}{2}+4\pi}{3}i} = \sqrt[3]{2}e^{\frac{9\pi}{6}i} = \sqrt[3]{2}e^{\frac{3\pi}{2}i} = \sqrt[3]{2}\cos\frac{3\pi}{2} + \sqrt[3]{2}\left(\sin\frac{3\pi}{2}\right)i = -\sqrt[3]{2}i.$$

   The equation $x^3 = 2i$ has three solutions:

   $$\frac{\sqrt[3]{2}\times\sqrt{3}}{2} + \frac{\sqrt[3]{2}}{2}i \quad ; \quad \frac{-\sqrt[3]{2}\times\sqrt{3}}{2} + \frac{\sqrt[3]{2}}{2}i \quad ; \quad -\sqrt[3]{2}i.$$

2. $\begin{cases} z_1 + z_2 = 3+i \\ z_1 z_2 = 2. \end{cases}$

   Answer:

   We know from the course that z_1 and z_2 are solutions of this system if and only if they are the two roots of the quadratic polynomial

   $$z^2 - (3+i)z + 2.$$

To compute its roots, let us compute its discriminant:

$$\Delta = (3+i)^2 - 4 \times 1 \times 2 = 3^2 + i^2 + 2 \times 3 \times i - 8 = 9 - 1 + 6i - 8 = 6i.$$

Then the polar form of Δ is

$$\Delta = 6e^{\frac{\pi}{2}i}$$

and once again from the course it admits two square roots

$$\delta_0 = \sqrt{6}e^{\frac{\frac{\pi}{2}}{2}i} = \sqrt{6}e^{\frac{\pi}{4}i} = \sqrt{6}\cos\frac{\pi}{4} + \sqrt{6}\left(\sin\frac{\pi}{4}\right)i = \sqrt{3} + \sqrt{3}i$$

and

$$\delta_1 = \sqrt{6}e^{\frac{\frac{\pi}{2}+2\pi}{2}i} = \sqrt{6}e^{\frac{5\pi}{4}i} = \sqrt{6}\cos\frac{5\pi}{4} + \sqrt{6}\left(\sin\frac{5\pi}{4}\right)i = -\sqrt{3} - \sqrt{3}i.$$

So the roots of the polynomial are

$$z = \frac{3+i+\delta_0}{2} = \frac{(3+\sqrt{3}) + (1+\sqrt{3})i}{2}$$
$$\text{and} \quad z' = \frac{3+i+\delta_1}{2} = \frac{(3-\sqrt{3}) + (1-\sqrt{3})i}{2}.$$

Finally, the original system of equations has two solutions:

$$\begin{cases} z_1 = \frac{(3+\sqrt{3})+(1+\sqrt{3})i}{2} \\ z_2 = \frac{(3-\sqrt{3})+(1-\sqrt{3})i}{2} \end{cases} \quad \text{and} \quad \begin{cases} z_1 = \frac{(3-\sqrt{3})+(1-\sqrt{3})i}{2} \\ z_2 = \frac{(3+\sqrt{3})+(1+\sqrt{3})i}{2} \end{cases}$$

Exercise 4 (20%) : Compute:

1. $\lim_{z \to i} \frac{z-i}{z^2+(1-2i)z-1-i}$.

   Answer:

   First of all, remark that

   $$\lim_{z \to i} z - i = 0 \quad \text{and} \quad \lim_{z \to i} z^2 + (1-2i)z - 1 - i = 0$$

   so we can not compute the limit directly. But because of that, it is easy to factorize the denominator:

   $$z^2 + (1-2i)z - 1 - i = (z-i)(z+1-i)$$

   so

   $$\lim_{z \to i} \frac{z-i}{z^2+(1-2i)z-1-i} = \lim_{z \to i} \frac{z-i}{(z-i)(z+1-i)} = \lim_{z \to i} \frac{1}{z+1-i} = 1.$$

2. $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2i)^n}$.

Answer:

Remark that

$$\left|\frac{1}{2i}\right| = \frac{1}{|2i|} = \frac{1}{2} < 1$$

so from the course, we know that the series of term $(1/2i)^n$ converges to

$$\frac{1}{1-\frac{1}{2i}} = \frac{1}{\frac{2i-1}{2i}} = \frac{2i}{-1+2i} = \frac{2i(-1-2i)}{(-1+2i)(-1-2i)} = \frac{4-2i}{5}.$$

Finally,

$$\sum_{n=0}^{+\infty} \frac{1}{(2i)^n} = \frac{4-2i}{5}.$$

Exercise 5 (40%) : The aim of this exercise is to find a new formula for the square roots of a complex number $z_0 \in \mathbb{C}$. Let $z = a + bi \in \mathbb{C}$ (with $a, b \in \mathbb{R}$)

1. Give (and justify) a condition (system of equations) relating a, b, $\operatorname{Re} z_0$ and $\operatorname{Im} z_0$ which is equivalent to $z^2 = z_0$.

   Answer:

   As $z = a + bi$,

   $$z^2 = (a+bi)^2 = (a^2 - b^2) + 2abi$$

   where $a^2 - b^2$ and $2ab$ are real numbers. So we get

   $$z^2 = z_0 \Leftrightarrow \begin{cases} a^2 - b^2 = \operatorname{Re} z_0 \\ 2ab = \operatorname{Im} z_0. \end{cases}$$

2. We set $A = 2a^2$ and $B = -2b^2$. Prove that the previous condition implies that

   $$\begin{cases} A + B = 2\operatorname{Re} z_0 \\ AB = -(\operatorname{Im} z_0)^2 \\ A \geqslant 0 \\ B \leqslant 0. \end{cases}$$

   Answer:

   If the condition of question 1 is satisfied, we get

   $$A + B = 2a^2 - 2b^2 = 2\operatorname{Re} z_0.$$

   We also get

   $$AB = 2a^2 \times (-2b^2) = -4a^2b^2 = -(2ab)^2 = -(\operatorname{Im} z_0)^2.$$

   Moreover, as $a, b \in \mathbb{R}$, we have

   $$A = 2a^2 \geqslant 0 \quad \text{and} \quad B = -2b^2 \leqslant 0.$$

3. Find all the possible pairs (A, B) satisfying the previous condition in function of $\operatorname{Re} z_0$ and $|z_0|$ (justify).
Answer:
To find all these solutions, we start by solving the system of equations

$$\begin{cases} A+B = 2\operatorname{Re} z_0 \\ AB = -(\operatorname{Im} z_0)^2. \end{cases}$$

We know that A and B satisfy this system if and only if they are the roots of the polynomial

$$x^2 - 2(\operatorname{Re} z_0)x - (\operatorname{Im} z_0)^2.$$

To find the roots of this polynomial, we compute first its discriminant:

$$\Delta = (2\operatorname{Re} z_0)^2 - 4 \times 1 \times (-(\operatorname{Im} z_0)^2) = 4(\operatorname{Re} z_0)^2 + 4(\operatorname{Im} z_0)^2 = 4|z_0|^2$$

which is real and non negative. So we get that the two square roots of Δ are

$$\delta_1 = 2|z_0| \quad \text{and} \quad \delta_2 = -2|z_0|.$$

Finally, the roots of the polynomial are

$$x_1 = \frac{2\operatorname{Re} z_0 + 2|z_0|}{2} = \operatorname{Re} z_0 + |z_0| \quad \text{and} \quad x_2 = \frac{2\operatorname{Re} z_0 - 2|z_0|}{2} = \operatorname{Re} z_0 - |z_0|.$$

From that, we deduce that the solutions of

$$\begin{cases} A+B = 2\operatorname{Re} z_0 \\ AB = -(\operatorname{Im} z_0)^2 \end{cases}$$

are

$$\begin{cases} A = \operatorname{Re} z_0 + |z_0| \\ B = \operatorname{Re} z_0 - |z_0| \end{cases} \quad \text{and} \quad \begin{cases} A = \operatorname{Re} z_0 - |z_0| \\ B = \operatorname{Re} z_0 + |z_0| \end{cases}$$

Remark that $|z_0|^2 = (\operatorname{Re} z_0)^2 + (\operatorname{Im} z_0)^2 \geqslant (\operatorname{Re} z_0)^2$ so, taking the positive square roots, $|z_0| \geqslant |\operatorname{Re} z_0|$. Thus,

- if $\operatorname{Re} z_0 \leqslant 0$, $\operatorname{Re} z_0 - |z_0| \leqslant 0$ and $\operatorname{Re} z_0 + |z_0| = |z_0| - |\operatorname{Re} z_0| \geqslant 0$;
- if $\operatorname{Re} z_0 \geqslant 0$, $\operatorname{Re} z_0 - |z_0| = |\operatorname{Re} z_0| - |z_0| \leqslant 0$ and $\operatorname{Re} z_0 + |z_0| \geqslant 0$.

Finally, in any case, $\operatorname{Re} z_0 - |z_0| \leqslant 0$ and $\operatorname{Re} z_0 + |z_0| \geqslant 0$ so if we add the conditions $A \geqslant 0$ and $B \leqslant 0$, the only pair (A, B) satisfying the condition of question 2 is

$$\begin{cases} A = \operatorname{Re} z_0 + |z_0| \\ B = \operatorname{Re} z_0 - |z_0| \end{cases}$$

4. Deduce a formula giving the square roots of z_0 (different from the one in the course).
Note: the formula can be defined by cases, depending on the sign of $\operatorname{Im} z_0$.
Answer:
From the definition of A and B, we have

$$a^2 = \frac{A}{2} = \frac{|z_0| + \operatorname{Re} z_0}{2} \quad \text{and} \quad b^2 = \frac{-B}{2} = \frac{|z_0| - \operatorname{Re} z_0}{2}$$

As these two numbers are positive, it permits to find four possibilities for a and b:

$$\left\{\begin{array}{l} a_1 = \sqrt{\frac{|z_0|+\operatorname{Re} z_0}{2}} \\ b_1 = \sqrt{\frac{|z_0|-\operatorname{Re} z_0}{2}} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} a_2 = \sqrt{\frac{|z_0|+\operatorname{Re} z_0}{2}} \\ b_2 = -\sqrt{\frac{|z_0|-\operatorname{Re} z_0}{2}} \end{array}\right.$$

$$\text{and} \quad \left\{\begin{array}{l} a_3 = -\sqrt{\frac{|z_0|+\operatorname{Re} z_0}{2}} \\ b_3 = \sqrt{\frac{|z_0|-\operatorname{Re} z_0}{2}} \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l} a_4 = -\sqrt{\frac{|z_0|+\operatorname{Re} z_0}{2}} \\ b_4 = -\sqrt{\frac{|z_0|-\operatorname{Re} z_0}{2}}. \end{array}\right.$$

At the second question, we only proved an implication. So it is not automatic that all these pairs gives square roots of z_0 (it is even impossible as we know that z_0 has one or two square roots). More precisely, the condition of the first question which is not automatically satisfied is $2ab = \operatorname{Im} z_0$. Remark that

$$\begin{aligned} 2\sqrt{\frac{|z_0| + \operatorname{Re} z_0}{2}}\sqrt{\frac{|z_0| - \operatorname{Re} z_0}{2}} &= \frac{2}{2}\sqrt{(|z_0| + \operatorname{Re} z_0)(|z_0| - \operatorname{Re} z_0)} \\ &= \sqrt{|z_0|^2 - (\operatorname{Re} z_0)^2} = \sqrt{(\operatorname{Im} z_0)^2} = |\operatorname{Im} z_0| \end{aligned}$$

and as a consequence $2a_1b_1 = |\operatorname{Im} z_0|$, $2a_2b_2 = -|\operatorname{Im} z_0|$, $2a_3b_3 = -|\operatorname{Im} z_0|$ and $2a_4b_4 = |\operatorname{Im} z_0|$. So

- If $\operatorname{Im} z_0 \geqslant 0$, the first and the fourth possibilities work and we get the two square roots of z_0

$$\sqrt{\frac{|z_0| + \operatorname{Re} z_0}{2}} + \sqrt{\frac{|z_0| - \operatorname{Re} z_0}{2}}i \quad \text{and} \quad -\sqrt{\frac{|z_0| + \operatorname{Re} z_0}{2}} - \sqrt{\frac{|z_0| - \operatorname{Re} z_0}{2}}i.$$

- If $\operatorname{Im} z_0 \leqslant 0$, the second and the third possibilities work and we get the two square roots of z_0

$$\sqrt{\frac{|z_0| + \operatorname{Re} z_0}{2}} - \sqrt{\frac{|z_0| - \operatorname{Re} z_0}{2}}i \quad \text{and} \quad -\sqrt{\frac{|z_0| + \operatorname{Re} z_0}{2}} + \sqrt{\frac{|z_0| - \operatorname{Re} z_0}{2}}i.$$

Notice that if $\operatorname{Im} z_0 = 0$, then $|z_0| = |\operatorname{Re} z_0|$ and the two cases coincide.

5. Compute the square roots of $3 + 4i$.
Answer:
First of all, $|3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. As $\operatorname{Im}(3 + 4i) = 4 \geqslant 0$, we apply the first formula of the previous question and the square roots of $3 + 4i$ are

$$\sqrt{\frac{|3 + 4i| + \operatorname{Re}(3 + 4i)}{2}} + \sqrt{\frac{|3 + 4i| - \operatorname{Re}(3 + 4i)}{2}}i = \sqrt{\frac{5 + 3}{2}} + \sqrt{\frac{5 - 3}{2}}i = 2 + i$$

and

$$-\sqrt{\frac{|3 + 4i| + \operatorname{Re}(3 + 4i)}{2}} - \sqrt{\frac{|3 + 4i| - \operatorname{Re}(3 + 4i)}{2}}i = -2 - i.$$

Name: (to hand in)

Complex Analysis - Mid-term exam - Answers
Exercise 2

The number $z \in \mathbb{C}$ is represented in the following diagram. Construct the point representing

$$2e^{-\frac{2\pi}{3}i} \times \overline{z} + i$$

on the diagram. Justify the construction geometrically by drawing some other points and marking clearly the steps of the construction.