Complex Analysis - Mid-term exam

Duration: 90 minutes

Documents and electronic devices are forbidden.

Give the answers in algebraic form (*i.e.* $a + bi$ with $a, b \in \mathbb{R}$).

All the answers should be properly justified and explained (any student who attended the course should be able to understand). **An important part of the grade will reward the exactness and rigor of reasoning (about 60%).**

The percentage of the grade awarded on each exercise is specified between parenthesis.

Exercise 1 (10%) : Compute $\operatorname{Arg}\left(-5 + 5\sqrt{3}i\right)$.

Exercise 2 (10%) : See other sheet.

Exercise 3 (20%) : Solve the equations or system of equations (in $\mathbb{C}$):

1. $x^3 = 2i$.
2. $\begin{cases} z_1 + z_2 = 3 + i \\ z_1 z_2 = 2. \end{cases}$

Exercise 4 (20%) : Compute:

1. $\displaystyle\lim_{z \to i} \frac{z - i}{z^2 + (1 - 2i)z - 1 - i}$.
2. $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{(2i)^n}$.

Exercise 5 (40%) : The aim of this exercise is to find a new formula for the square roots of a complex number $z_0 \in \mathbb{C}$. Let $z = a + bi \in \mathbb{C}$ (with $a, b \in \mathbb{R}$)

1. Give (<u>and justify</u>) a condition (system of equations) relating a, b, $\operatorname{Re} z_0$ and $\operatorname{Im} z_0$ which is <u>equivalent</u> to $z^2 = z_0$.

2. We set $A = 2a^2$ and $B = -2b^2$. <u>Prove</u> that the previous condition implies that

$$\begin{cases} A + B = 2 \operatorname{Re} z_0 \\ AB = -(\operatorname{Im} z_0)^2 \\ A \geqslant 0 \\ B \leqslant 0. \end{cases}$$

3. Find all the possible pairs (A, B) satisfying the previous condition in function of $\operatorname{Re} z_0$ and $|z_0|$ (<u>justify</u>).

4. Deduce a formula giving the square roots of z_0 (different from the one in the course).

   <u>Note:</u> the formula can be defined by cases, depending on the sign of $\operatorname{Im} z_0$.

5. Compute the square roots of $3 + 4i$.

Name: (to hand in)

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Exercise 2

The number $z \in \mathbb{C}$ is represented in the following diagram. Construct the point representing

$$2e^{-\frac{2\pi}{3}i} \times \overline{z} + i$$

on the diagram. Justify the construction geometrically by drawing some other points and marking clearly the steps of the construction.