Complex Analysis - Quiz 9
Solution

1. Justify that, for a curve γ of $\mathbb{C}$, $\gamma + (-\gamma)$ is a closed curve.

   Recall that $-\gamma$ is the curve γ travelled in the opposite direction. Therefore, $-\gamma$ starts where γ finishes and we can concatenate these two curves. The starting point of $\gamma + (-\gamma)$ is the starting point of γ, and its ending point is the ending point of $-\gamma$, that is the starting point of γ also. Finally, $\gamma + (-\gamma)$ starts and ends at the same points: it is a closed curve.

2. For $z_0 \in \mathbb{C}$ that is not on γ, compute $I(\gamma + (-\gamma); z_0)$.

   By definition of the index of a curve with respect to a point, and using results of the course about integrals along concatenations of curves and the opposite of a curve, we have

$$\begin{aligned} I(\gamma + (-\gamma); z_0) &= \frac{1}{2\pi i} \int_{\gamma + (-\gamma)} \frac{1}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi i} \left(\int_{\gamma} \frac{1}{z - z_0} \, \mathrm{d}z + \int_{-\gamma} \frac{1}{z - z_0} \, \mathrm{d}z \right) \\ &= \frac{1}{2\pi i} \left(\int_{\gamma} \frac{1}{z - z_0} \, \mathrm{d}z - \int_{\gamma} \frac{1}{z - z_0} \, \mathrm{d}z \right) = 0. \end{aligned}$$

3. Interpret this result geometrically.

   It means that z_0 is never inside the closed curve $\gamma + (-\gamma)$. This is geometrically obvious as $\gamma + (-\gamma)$ consists to travel one direction and the opposite direction along the same trajectory, enclosing none of $\mathbb{C}$.