Complex Analysis - Midterm examination

Duration: 90 minutes (2018/11/26).

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (12) Solve the following equations (find **all** solutions in $\mathbb{C}$).

1. $z^2 - 2z + 1 - 2i = 0$.
2. $\exp(z/2) = i + 1$.
3. $z^7 = -i$.

Problem 2 : (8) We consider the complex numbers $z_1 = 1 + i\sqrt{3}$, $z_2 = 1 + i$ and $z_3 = z_1/z_2$.

1. Write z_1, z_2 and z_3 in polar form.
2. Deduce the exact values of $\cos(\pi/12)$ and $\sin(\pi/12)$.

Problem 3 : (12) For each of the following functions:

- give its maximal domain;
- tell where it is analytic;
- computes its derivative on the set where it is analytic.

All answer should be justified carefully (all necessary theorems should be cited).

1. $f(z) = \exp(\cos(z))$.
2. $g(z) = \overline{z}(\overline{z} + \alpha + \beta i)$ where $\alpha, \beta \in \mathbb{R}$.
3. $h(z) = |\operatorname{Re} z| - |\operatorname{Im} z| i$.

Problem 4 : (8) Consider an analytic function $f : \mathbb{C} \to \mathbb{C}$ whose image is entirely contained in the unit circle (in other terms, $|f(z)| = 1$ for every $z \in \mathbb{C}$). Denote $u(x, y) = \operatorname{Re} f(x + yi)$ and $v(x, y) = \operatorname{Im} f(x + yi)$.

1. Compute $\dfrac{\partial |f(x+yi)|^2}{\partial x}$ and $\dfrac{\partial |f(x+yi)|^2}{\partial y}$ in terms of u, v and their partial derivatives.
2. Deduce that f is constant.