Complex Analysis - Midterm examination

Duration: 90 minutes (2017/11/20).

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (10) Solve the following equations (find **all** solutions in $\mathbb{C}$).

1. $z^5 = i - 1$.
2. $z^2 + iz + 1 + 3i = 0$.
3. $\exp(z+1) = 2i$.

Problem 2 : (7) Consider three distinct complex numbers z_1, z_2 and z_3. Denote by A_1, A_2 and A_3 the corresponding points of the plane. Consider the complex number

$$\omega = \frac{z_2 - z_1}{z_3 - z_1}.$$

Let $\omega = r\exp(\theta i)$ be the polar form of ω.

1. Describe geometrically the map f defined by $f(z) = \omega z$ in terms of r and θ.
2. Interpret r and θ in terms of the sides and the angles of the triangle $A_1A_2A_3$.
3. Deduce the possible values of ω when the triangle $A_1A_2A_3$ is equilateral.

Problem 3 : (10) For each of the following functions:

- give its maximal domain;
- tell where it is analytic;
- computes its derivative on the set where it is analytic.

All answer should be justified carefully (all necessary theorems should be cited).

1. $f(z) = \mathrm{Re}(z)^2 - 2\overline{z}$.
2. $g(z) = \begin{cases} \dfrac{\exp(z) - 1}{z} & \text{if } z \neq 0; \\ 1 & \text{if } z = 0. \end{cases}$

   Hint: You can use *L'Hôpital's rule* without proving it for complex functions if necessary.

Problem 4 : (6) We consider an entire function $f : \mathbb{C} \to \mathbb{C}$ (analytic on whole $\mathbb{C}$).

1. Prove that the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \overline{f(\overline{z})}$ is also entire.
2. Consider the function $h : \mathbb{C} \to \mathbb{C}$ defined by $h(z) = \overline{f(z)}$. Prove that if h is also entire, then f is constant.

Problem 5 : (7) We consider a continuous map $\varphi : \mathbb{C} \to \mathbb{C}$ satisfying, for any $z_1, z_2 \in \mathbb{C}$, $\varphi(z_1 + z_2) = \varphi(z_1) + \varphi(z_2)$ and $\varphi(z_1 z_2) = \varphi(z_1)\varphi(z_2)$.

1. Prove that $\varphi(1) = 0$ or $\varphi(1) = 1$.
2. We suppose that $\varphi(1) = 0$. Compute $\varphi(z)$ for all $z \in \mathbb{C}$.
3. We suppose instead that $\varphi(1) = 1$.
   (a) Prove that for any $n \in \mathbb{Z}$, $\varphi(n) = n$.
   (b) Prove that for any $r \in \mathbb{Q}$, $\varphi(r) = r$.
   (c) Prove that for any $x \in \mathbb{R}$, $\varphi(x) = x$.
4. Give formulas for all possible φ satisfying the above conditions.