Complex Analysis - Midterm examination
Solutions

Problem 1 : (8) Let $z_0 = 3 - \sqrt{3}i$.

1. Express z_0 in polar form (in the form $r\exp(\theta i)$).

   First of all $|z_0| = \sqrt{3^2 + (-\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$. Then, we have

   $$\frac{z_0}{|z_0|} = \frac{\sqrt{3}}{2} - \frac{1}{2}i = \exp(11\pi i/6)$$

   so the polar form of z_0 is $z_0 = 2\sqrt{3}\exp(11\pi i/6)$.

2. Solve the equation $z^3 = z_0$.

   For $z^3 = z_0$, we have $|z|^3 = |z_0|$ hence $|z| = \sqrt[3]{2\sqrt{3}} = \sqrt[6]{12}$. Moreover, we should have $3\operatorname{Arg}(z) = \operatorname{Arg}(z_0)$ up to a multiple of 2π hence $\operatorname{Arg}(z) = \operatorname{Arg}(z_0)/3 = 11\pi/18$ up to a multiple of $2\pi/3$. Hence, solutions of $z^3 = z_0$ are of the form

   $$z_k = \sqrt[6]{12}\exp((11 + 12k)\pi i/18)$$

   for $k \in \mathbb{Z}$. As $z_{k+3} = z_k$, the solutions are actually z_0, z_1 and z_2 (which are clearly distinct).

Problem 2 : (8) See other page.

Problem 3 : (8) Solve the equations (find **all** solutions in $\mathbb{C}$):

1. $z^2 - 3z + (3 - i) = 0$.

   We rewrite the equation as $(z-3/2)^2 - 9/4 + (3-i) = 0$ hence $(z-3/2)^2 = -3/4+i$. Let us find a square root of $-3/4 + i$. Such a solution can be written $\delta = \alpha + \beta i$ with $\alpha, \beta \in \mathbb{R}$ and we get

   $$\alpha^2 - \beta^2 = -3/4 \quad (1)$$
   $$2\alpha\beta = 1 \quad (2)$$

   then the sum of the squares of (1) and (2) gives

   $$\left(-\frac{3}{4}\right)^2 + 1^2 = (\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = \alpha^4 + 2\alpha^2\beta^2 + \beta^4 = (\alpha^2 + \beta^2)^2$$

   hence, as $\alpha^2 + \beta^2 \geqslant 0$,

   $$\alpha^2 + \beta^2 = \sqrt{\frac{9+16}{16}} = \frac{5}{4} \quad (3)$$

   so, by summing and computing the difference of (1) and (3): $\alpha^2 = 1/4$ and $\beta^2 = 1$. As $2\alpha\beta > 0$, α and β have the same sign. Hence, we get the two square roots

$\delta = 1/2 + i$ and $-\delta = -1/2 - i$. Notice that it is easy to check that actually $\delta^2 = -3/4 + i$, but we do not really need it as we know that there are exactly two square roots.

Finally, we get $z - 3/2 = \pm(1/2 + i)$ hence the two solutions of our equation are $z_1 = 2 + i$ and $z_2 = 1 - i$.

2. $\exp(-2z) = -1 + i$.

   To solve this equation, we write in polar coordinates $-1+i$. We have $|-1+i| = \sqrt{2}$ and $\operatorname{Arg}(-1+i) = \operatorname{Arg}(-1/\sqrt{2} + i/\sqrt{2}) = 3\pi/4$ up to a multiple of 2π. Hence, the logarithms of $-1+i$ are $\log(\sqrt{2}) + (3+8k)\pi i/4 = \log(2)/2 + (3+8k)\pi i/4$ for $k \in \mathbb{Z}$. So the initial equation is equivalent to $-2z = \log(2)/2 + (3+8k)\pi i/4$ so the solutions are $z_k = -\log(2)/4 - (3+8k)\pi i/8$ for $k \in \mathbb{Z}$ (they are all distinct).

Problem 4: (8) For each of the following functions, tell where it is analytic and, on this set, compute its derivative. In both cases justify precisely your answer.

1. $f(z) = \dfrac{\exp(3z-2)}{z-1}$ for $z \in \mathbb{C} \setminus \{1\}$;

   The numerator is entire as it is the composition of *exp* with a polynomial. The denominator is entire as it is a polynomial. Hence f is differentiable whenever the denominator does not vanish, that is when $z \neq 1$. We compute using the usual rules:

$$f'(z) = \frac{\frac{\mathrm{d}\exp(3z-2)}{\mathrm{d}z}(z-1) - \exp(3z-2)\frac{\mathrm{d}\,z-1}{\mathrm{d}z}}{(z-1)^2} = \frac{(3z-4)\exp(3z-2)}{(z-1)^2}.$$

2. $g(z) = \operatorname{Im}(z) + |\operatorname{Re}(z)|\, i$ for $z \in \mathbb{C}$.

   We have $g^{\mathbb{R}}(x,y) = (y, |x|)$. In particular, at $x_0 = 0$, $\partial g_2^{\mathbb{R}}/\partial x$ is not define, hence the function is not differentiable for $\operatorname{Re} z = 0$. For other values of (x,y), we have the following Jacobian matrix:

$$D_{g^{\mathbb{R}}}(x,y) = \begin{bmatrix} 0 & 1 \\ \operatorname{sgn}(x) & 0 \end{bmatrix}$$

   where $\operatorname{sgn}(x)$ is 1 for $x > 0$ and -1 for $x < 0$. For $x \neq 0$, the coefficients are continuous, hence $g^{\mathbb{R}}$ is differentiable. For g to be differentiable, we need the Cauchy-Riemann equations to be satisfied, hence $0 = 0$ and $1 = -\operatorname{sgn}(x)$. So g is differentiable whenever $x < 0$ *i.e.* whenever $\operatorname{Re}(z) < 0$ and, again by Cauchy-Riemann theorem, $g'(x) = -i$ in this case. Notice that actually $g(z) = -iz$ if $\operatorname{Re}(z) < 0$ and $g(z) = i\bar{z}$ if $\operatorname{Re}(z) > 0$.

Problem 5: (8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is infinitely many times differentiable. We say that f is *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on whole of $\mathbb{R}^2$. The aim of this exercise is to prove that harmonic functions are exactly functions that are real parts of entire functions from $\mathbb{C}$ to $\mathbb{C}$ (here, *entire* means functions that are infinitely many times differentiable on whole of $\mathbb{C}$).

1. We suppose that $h : \mathbb{C} \to \mathbb{C}$ is infinitely many times differentiable on whole of $\mathbb{C}$. We write $h^{\mathbb{R}} = (h_1, h_2)$ where $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$. Prove that h_1 is harmonic.

   Hint: Use Cauchy-Riemann equations and $\partial^2 h_1/\partial x\partial y = \partial^2 h_1/\partial y\partial x$ (Calculus 2).

   If h is infinitely many times differentiable on whole of $\mathbb{C}$, we have that h_1 and h_2 are infinitely many times differentiable and, by Cauchy-Riemann theorem,

$$\frac{\partial h_1}{\partial x} = \frac{\partial h_2}{\partial y} \quad \text{and} \quad \frac{\partial h_1}{\partial y} = -\frac{\partial h_2}{\partial x}.$$

   If we differentiate the first one with respect to x and the second one with respect to y, we obtain

$$\frac{\partial^2 h_1}{\partial x^2} = \frac{\partial^2 h_2}{\partial x\partial y} \quad \text{and} \quad \frac{\partial^2 h_1}{\partial y^2} = -\frac{\partial^2 h_2}{\partial y\partial x} = -\frac{\partial^2 h_2}{\partial x\partial y}$$

   (the last equality is true because these partial derivatives are continuous) hence by adding up:

$$\frac{\partial^2 h_1}{\partial x^2} + \frac{\partial^2 h_1}{\partial y^2} = 0$$

   so h_1 is harmonic.

2. We fix a harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$. Write the two differential equations

$$\text{(E1)}\ \frac{\partial g}{\partial y} = \dots \qquad\qquad \text{(E2)}\ \frac{\partial g}{\partial x} = \dots$$

   that $g : \mathbb{R}^2 \to \mathbb{R}$ must satisfy for $h(x + yi) = f(x, y) + g(x, y)i$ to be analytic on $\mathbb{C}$.

   If h is analytic, f and g satisfy the Cauchy-Riemann equations:

$$\text{(E1)}\ \frac{\partial g}{\partial y} = \frac{\partial f}{\partial x} \qquad\qquad \text{(E2)}\ \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial y}.$$

3. Prove that (E1) has a least a solution g_0.

   Let us write $G(x, y) = (\partial f/\partial x)(x, y)$. Then we can for example get a solution of (E1) by putting

$$g_0(x, y) = \int_0^y G(x, t)\,\mathrm{d}\,t,$$

   using the fundamental theorem of Calculus (over $\mathbb{R}$ as y is a real variable). It is infinitely times differentiable.

4. Prove that

$$\frac{\partial g_0}{\partial x} + \frac{\partial f}{\partial y}$$

   has the form $u(x)$ (does not depend on y).

   Let us differentiate this sum with respect to y:

$$
\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{\partial g_0}{\partial x}+\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 g_0}{\partial y \partial x}+\frac{\partial^2 f}{\partial y^2} \\
&= \frac{\partial}{\partial x}\frac{\partial g_0}{\partial y}+\frac{\partial^2 f}{\partial y^2} && \text{because } g_0 \text{ is twice differentiable} \\
&= \frac{\partial}{\partial x}\frac{\partial f}{\partial x}+\frac{\partial^2 f}{\partial y^2} && \text{by (E1)} \\
&= 0 && \text{because } f \text{ is harmonic}
\end{aligned}
$$

so the sum does not depend on y. It is a function u of x.

Let $U : \mathbb{R} \to \mathbb{R}$ be an anti-derivative of u and fix $g(x,y) = g_0(x,y) - U(x)$.

5. Prove that $h(x+yi) = f(x,y) + g(x,y)i$ is analytic on $\mathbb{C}$.

   We need to check (E1) and (E2). For (E1):

$$
\frac{\partial g}{\partial y} = \frac{\partial g_0}{\partial y} - \frac{\partial U}{\partial y} = \frac{\partial g_0}{\partial y} = \frac{\partial f}{\partial x}
$$

   because U does not depend on y and g_0 has been chosen to be a solution of (E1).

   For (E2):

$$
\frac{\partial g}{\partial x} = \frac{\partial g_0}{\partial x} - \frac{\partial U}{\partial x} = \frac{\partial g_0}{\partial x} - u(x) = -\frac{\partial f}{\partial y}
$$

   as $u(x) = \partial g_0/\partial x + \partial f/\partial y$. As both Cauchy-Riemann equations are satisfied and both f and g are infinitely many times differentiable, $h(x+yi) = f(x,y) + g(x,y)i$ is analytic on $\mathbb{C}$.

   Notice that actually, the g is unique up to an additive constant.

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Problem 2

The number $z \in \mathbb{C}$ is represented in the following figure. Construct the point representing

$$\overline{z\exp(3\pi i/4)} + 2 + i$$

on the figure. Justify the construction geometrically by drawing some other points and marking clearly the steps of the construction.