Complex Analysis - Midterm examination

Duration: 90 minutes (2016/11/14).

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (8) Let $z_0 = 3 - \sqrt{3}i$.

1. Express z_0 in polar form (in the form $r\exp(\theta i)$).
2. Solve the equation $z^3 = z_0$.

Problem 2 : (8) See other page.

Problem 3 : (8) Solve the equations (find **all** solutions in $\mathbb{C}$):

1. $z^2 - 3z + (3 - i) = 0$.
2. $\exp(-2z) = -1 + i$.

Problem 4 : (8) For each of the following functions, tell where it is analytic and, on this set, compute its derivative. In both cases justify precisely your answer.

1. $f(z) = \dfrac{\exp(3z-2)}{z-1}$ for $z \in \mathbb{C} \setminus \{1\}$;
2. $g(z) = \mathrm{Im}(z) + |\mathrm{Re}(z)|\, i$ for $z \in \mathbb{C}$.

Problem 5 : (8) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function that is infinitely many times differentiable. We say that f is *harmonic* if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

on whole of $\mathbb{R}^2$. The aim of this exercise is to prove that harmonic functions are exactly functions that are real parts of entire functions from $\mathbb{C}$ to $\mathbb{C}$ (here, *entire* means functions that are infinitely many times differentiable on whole of $\mathbb{C}$).

1. We suppose that $h : \mathbb{C} \to \mathbb{C}$ is infinitely many times differentiable on whole of $\mathbb{C}$. We write $h^{\mathbb{R}} = (h_1, h_2)$ where $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$. Prove that h_1 is harmonic.

   Hint: Use Cauchy-Riemann equations and $\partial^2 h_1/\partial x \partial y = \partial^2 h_1/\partial y \partial x$ (Calculus 2).

2. We fix a harmonic function $f : \mathbb{R}^2 \to \mathbb{R}$. Write the two differential equations

$$\text{(E1)}\ \frac{\partial g}{\partial y} = \dots \qquad\qquad \text{(E2)}\ \frac{\partial g}{\partial x} = \dots$$

   that $g : \mathbb{R}^2 \to \mathbb{R}$ must satisfy for $h(x+yi) = f(x,y) + g(x,y)i$ to be analytic on $\mathbb{C}$.

3. Prove that (E1) has a least a solution g_0.

4. Prove that

$$\frac{\partial g_0}{\partial x} + \frac{\partial f}{\partial y}$$

   has the form $u(x)$ (does not depend on y).

   Let $U : \mathbb{R} \to \mathbb{R}$ be an anti-derivative of u and fix $g(x,y) = g_0(x,y) - U(x)$.

5. Prove that $h(x+yi) = f(x,y) + g(x,y)i$ is analytic on $\mathbb{C}$.

Name: (to hand in)

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Problem 2

The number $z \in \mathbb{C}$ is represented in the following figure. Construct the point representing

$$\overline{z\exp(3\pi i/4)} + 2 + i$$

on the figure. Justify the construction geometrically by drawing some other points and marking clearly the steps of the construction.