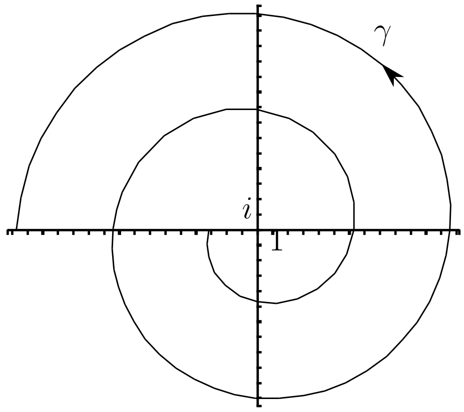
Complex Analysis - 忘年 homework - solutions

Problem 1 : For each of the following cases, draw the curve γ and compute the integral of f along γ:

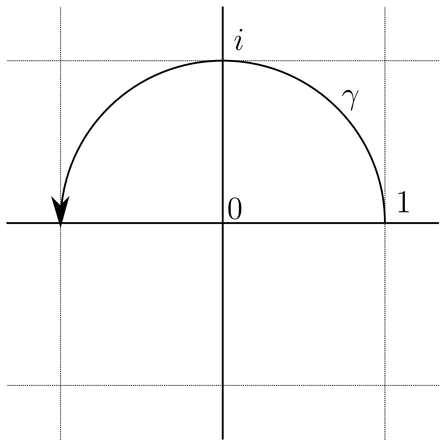
1. $f(z) = z^3$ along $\gamma(\theta) = \theta e^{i\theta}$ for $\theta \in [\pi, 5\pi]$;



The function $f(z)$ admits an antiderivative $F(z) = z^4/4$ on whole $\mathbb{C}$. Therefore, by the fundamental theorem of calculus,

$$\int_\gamma z^3 \,\mathrm{d}z = F(\gamma(5\pi)) - F(\gamma(\pi)) = \frac{(5\pi\exp(i5\pi))^4 - (\pi\exp(i\pi))^4}{4}$$
$$= \frac{625\pi^4\exp(20i\pi) - \pi^4\exp(4i\pi)}{4} = 156\pi^4.$$

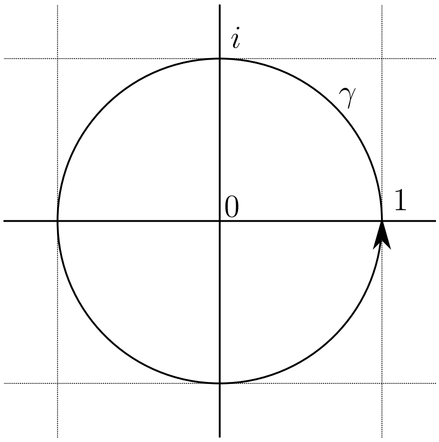
2. $f(z) = z\exp(z^2)$ along $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, \pi]$;



The function $f(z)$ admits an antiderivative $F(z) = \exp(z^2)/2$ on whole $\mathbb{C}$. Therefore, by the fundamental theorem of calculus,

$$\int_\gamma z\exp(z^2)\,\mathrm{d}z = F(\gamma(\pi)) - F(\gamma(0)) = \frac{\exp((-1)^2) - \exp(1^2)}{2} = 0.$$

3. $f(z) = \dfrac{1}{z^2}$ along $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$;



By definition,

$$\int_\gamma \frac{1}{z^2}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\gamma(\theta)^2}\gamma'(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1}{\exp(2i\theta)} i\exp(i\theta)\,\mathrm{d}\theta = \int_0^{2\pi} i\exp(-i\theta)\,\mathrm{d}\theta$$
$$= \left[\frac{i}{-i}\exp(-i\theta)\right]_0^{2\pi} = 0.$$

4. $f(z) = \overline{z}$ along $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Does f satisfy Cauchy theorem? Why?

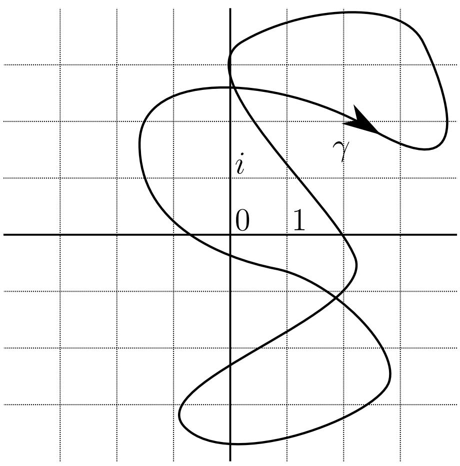
Same curve as in question 3.

By definition,

$$\int_\gamma \overline{z}\,\mathrm{d}z = \int_0^{2\pi} \overline{\gamma(\theta)}\gamma'(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} \overline{e^{i\theta}} i e^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} e^{-i\theta} i e^{i\theta}\,\mathrm{d}\theta = 2\pi i.$$

It does not satisfy Cauchy's theorem. The reason is that f is not analytic.

Problem 2 : We consider the following closed curve γ:



Evaluate the following integrals (justify):

1. $\displaystyle\int_\gamma \frac{1}{z}\,\mathrm{d}z$;

   The curve γ is homotopic in $\mathbb{C}\setminus\{0\}$ to the unit circle in clockwise direction. Therefore, $I(\gamma;0)=-1$. In other terms,
$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = -2\pi i.$$

2. $\displaystyle\int_\gamma \frac{5}{z-2+2i}\,\mathrm{d}z$;

   The curve γ is homotopic in $\mathbb{C}\setminus\{2-2i\}$ to the unit circle centered at $2-2i$ in anticlockwise direction. Therefore, $I(\gamma;2-2i)=1$. Finally,
$$\int_\gamma \frac{5}{z-2+2i}\,\mathrm{d}z = 5\times 2\pi i = 10\pi i.$$

3. $\displaystyle\int_\gamma \exp\frac{1}{z+2}\,\mathrm{d}z$;

   The point -2 does not lie inside the curve γ and $\exp(1/(z+2))$ is analytic on $\mathbb{C}\setminus\{-2\}$. Therefore, thanks to Cauchy's theorem,
$$\int_\gamma \exp\frac{1}{z+2}\,\mathrm{d}z = 0.$$

4. $\displaystyle\int_\gamma \frac{z-1}{z^2-2z+10}\,\mathrm{d}z$;

   Notice that for $z\in\mathbb{C}\setminus\{1-3i,1+3i\}$, we have
$$\frac{z-1}{z^2-2z+10} = \frac{z-1}{(z-1+3i)(z-1-3i)} = \frac{1/2}{z-1-3i} + \frac{1/2}{z-1+3i}$$
   so
$$\int_\gamma \frac{z-1}{z^2-2z+10}\,\mathrm{d}z = \int_\gamma \frac{1/2}{z-1-3i}\,\mathrm{d}z + \int_\gamma \frac{1/2}{z-1+3i}\,\mathrm{d}z.$$
   The curve γ is homotopic in $\mathbb{C}\setminus\{1+3i\}$ to the unit circle centered at $1+3i$ in anticlockwise direction. Therefore, $I(\gamma;1+3i)=1$. In the same way, the curve γ is homotopic in $\mathbb{C}\setminus\{1-3i\}$ to the unit circle centered at $1-3i$ in anticlockwise direction. Therefore, $I(\gamma;1-3i)=1$. Finally,
$$\int_\gamma \frac{z-1}{z^2-2z+10}\,\mathrm{d}z = \int_\gamma \frac{1/2}{z-1-3i}\,\mathrm{d}z + \int_\gamma \frac{1/2}{z-1+3i}\,\mathrm{d}z = \frac{1}{2}(2\pi i+2\pi i) = 2\pi i.$$

5. $\displaystyle\int_\gamma \frac{\sin(z)}{z-1}\,\mathrm{d}z.$

   The curve γ is homotopic in $\mathbb{C}\setminus\{1\}$ to the unit circle centered at 1 in clockwise direction. Therefore, $I(\gamma;1)=-1$. Moreover, Cauchy's integral formula gives, as sin is entire,
$$\sin(1)I(\gamma;1) = \frac{1}{2\pi i}\int_\gamma \frac{\sin(z)}{z-1}\,\mathrm{d}z$$

so

$$\int_\gamma \frac{\sin(z)}{z-1}\,\mathrm{d}z = 2\pi i \sin(1) I(\gamma;1) = -2\pi\sin(1)i.$$

Give the index of γ with respect to (justify):

6. $z_0 = 0$;

   We saw in question 1 that $I(\gamma;0) = -1$.

7. $z_1 = 2+i$;

   The point $2+i$ does not lie inside the curve γ. Therefore, $I(\gamma;2+i) = 0$.

8. $z_2 = 2-2i$.

   We saw in question 2 that $I(\gamma;2-2i) = 1$.

Problem 3 : The aim of this problem is to compute the following integral:

$$I = \int_0^{2\pi} \frac{\cos\theta}{2+\cos\theta}\,\mathrm{d}\theta.$$

1. Consider the curve $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0,2\pi]$. Compute

$$\int_\gamma \frac{z^2+1}{z^3+4z^2+z}\,\mathrm{d}z.$$

   Notice that for $z \in \mathbb{C} \setminus \{-2-\sqrt{3}, -2+\sqrt{3}, 0\}$, we have

$$\frac{z^2+1}{z^3+4z^2+z} = \frac{2/\sqrt{3}}{z+2+\sqrt{3}} - \frac{2/\sqrt{3}}{z+2-\sqrt{3}} + \frac{1}{z}$$

   so

$$\int_\gamma \frac{z^2+1}{z^3+4z^2+z}\,\mathrm{d}z = \int_\gamma \frac{2/\sqrt{3}}{z+2+\sqrt{3}}\,\mathrm{d}z - \int_\gamma \frac{2/\sqrt{3}}{z+2-\sqrt{3}}\,\mathrm{d}z + \int_\gamma \frac{1}{z}\,\mathrm{d}z.$$

   As $|2+\sqrt{3}| = 2+\sqrt{3} > 1$, $2+\sqrt{3}$ is not inside the unit circle and $I(\gamma;2+\sqrt{3}) = 0$. As $|-2+\sqrt{3}| = 2-\sqrt{3} < 1$, $-2+\sqrt{3}$ lies inside the unit circle and $I(\gamma;-2+\sqrt{3}) = 1$ (γ turns in the anticlockwise direction). In the same way $I(\gamma;0) = 1$. Thus, using Cauchy's integral theorem, we obtain

$$\begin{aligned}\int_\gamma \frac{z^2+1}{z^3+4z^2+z}\,\mathrm{d}z &= \int_\gamma \frac{2/\sqrt{3}}{z+2+\sqrt{3}}\,\mathrm{d}z - \int_\gamma \frac{2/\sqrt{3}}{z+2-\sqrt{3}}\,\mathrm{d}z + \int_\gamma \frac{1}{z}\,\mathrm{d}z \\ &= \left(-\frac{2}{\sqrt{3}}+1\right)2\pi i.\end{aligned}$$

2. Using the original definition of curve integrals, compute

$$\int_\gamma \frac{z+z^{-1}}{z^2+4z+1}\,\mathrm{d}z$$

as an integral of trigonometric functions.

By definition, we have

$$\begin{aligned}\int_\gamma \frac{z+z^{-1}}{z^2+4z+1}\,\mathrm{d}z &= \int_0^{2\pi} \frac{e^{i\theta}+(e^{i\theta})^{-1}}{(e^{i\theta})^2+4e^{i\theta}+1} ie^{i\theta}\,\mathrm{d}\theta = i\int_0^{2\pi} \frac{e^{i\theta}+e^{-i\theta}}{e^{i\theta}+4+e^{-i\theta}}\,\mathrm{d}\theta \\ &= i\int_0^{2\pi} \frac{2\cos\theta}{2\cos\theta+4}\,\mathrm{d}\theta = Ii.\end{aligned}$$

3. Deduce the value of I.

   Using the previous questions, we obtain $Ii = \left(-2/\sqrt{3}+1\right)2\pi i$ so

$$I = \left(-\frac{2}{\sqrt{3}}+1\right)2\pi.$$