Complex Analysis - Final examination

Duration: 90 minutes.

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

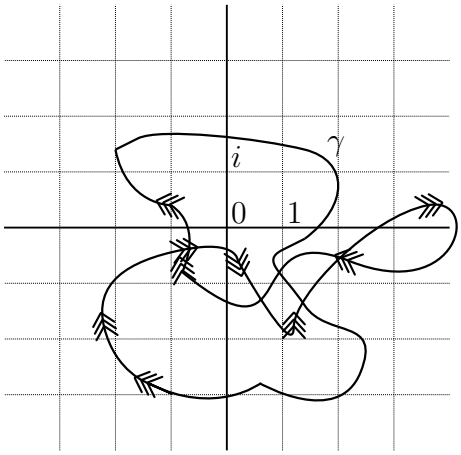
All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 50. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (15) For each of the following cases, **draw** the curve γ and compute the **integral** of f along γ. In each case, indicate which **theorem(s)** you use and check all the necessary **assumptions / hypotheses**.

1. $f(z) = z\cos(z)$ along $\gamma(t) = t^2 + it$ for $t \in [0, 1]$;
2. $f(z) = \mathrm{Im}(z) + \mathrm{Re}(z)i$ along $\gamma(t) = \exp(t) + ti$ for $t \in [0, 1]$;
3. $f(z) = \dfrac{\cos(z^2 - 1)}{z^2 - 1}$ along $\gamma(\theta) = 2e^{i\theta}$ for $\theta \in [0, 2\pi]$.

Problem 2 : (20) We consider the following closed curve γ:



Evaluate the following **integrals**:

1. $\displaystyle\int_\gamma \frac{1}{z-3}\,\mathrm{d}\,z$;
2. $\displaystyle\int_\gamma \cos\left(\frac{\exp(z+1)}{z+1}\right)\mathrm{d}\,z$;
3. $\displaystyle\int_\gamma \frac{2i-2}{z^2+2i}\,\mathrm{d}\,z$;
4. $\displaystyle\int_\gamma \frac{\exp(\sin(z)^2 - i)}{z+2i}\,\mathrm{d}\,z$.

In each case, indicate which **theorem(s)** you use and check all necessary **assumptions / hypotheses**.

Problem 3 : (15) Let U be a connected and simply-connected open subset of $\mathbb{C}$ (recall that *connected* means "in one piece" and *simply-connected* means that any closed curve in U is homotopic, in U, to a point). We consider an analytic function $f : U \to \mathbb{C} \setminus \{0\}$.

1. Prove that the function $h : U \to \mathbb{C}$ defined by $h(z) = f'(z)/f(z)$ is analytic.
2. Prove that h admits an antiderivative H on all of U (**justify carefully**).
3. Prove that the function $k : U \to \mathbb{C}$ defined by $k(z) = f(z)\exp(-H(z))$ is a non-zero constant.
4. Deduce that there exists an analytic function $g : U \to \mathbb{C}$ such that $\exp(g(z)) = f(z)$ for any $z \in \mathbb{C}$.
5. Find a counter-example to Question 4 when U is not simply connected (but all other assumptions are satisfied) (**justify carefully**). *Hint: g is a kind of logarithm.*