Review Exercises for Chapter 2

1. Evaluate the following integrals:

   (a) $\int_\gamma \sin z \, dz$, where γ is the unit circle

   (b) $\int_\gamma \frac{\sin z}{z} dz$, where γ is the unit circle

   (c) $\int_\gamma \frac{\sin z}{z^2} dz$, where γ is the unit circle

   (d) $\int_{|z|=1} \frac{\sin e^z}{z^2} dz$

2. Let f and g be analytic in a region A. Prove: If $|f| = |g|$ on A and $f \neq 0$ except at isolated points, then $f(z) = e^{i\theta} g(z)$ on A for some constant $\theta, 0 \leq \theta < 2\pi$.

3. Let f be analytic on $\{z$ such that $|z| > 1\}$. Show that if γ_r is the circle of radius $r > 1$ and center 0, then $\int_{\gamma_r} f$ is independent of r.

4.• Let $f(z) = P(z)/Q(z)$ where P and Q are polynomials and Q has a degree of at least 2 more than that of P.

   (a) Argue that if R is sufficiently large, there is a constant M such that

   $$\left| \frac{P(z)}{Q(z)} \right| \leq \frac{M}{|z|^2} \quad \text{for} \quad |z| \geq R.$$

   (b) If γ is a circle of radius r and center 0 with r large enough that f is analytic outside γ, prove that $\int_\gamma f(z) dz = 0$. *Hint*: Use Exercise 3 and let $r \to \infty$.

   (c) Evaluate $\int_\gamma \frac{dz}{1+z^2}$, where γ is a circle of radius 2 and center 0.

5. Evaluate $\int_\gamma f$, where $f(x+iy) = x^2 + iy^2$ and γ is the line joining 1 to i.

6. Suppose that $u : \mathbb{C} \to \mathbb{R}$ is a function that is harmonic on all of $\mathbb{C}$.

   (a) Show that if u is bounded above then it must be constant. *Hint*: Examine the solution to Worked Example 2.5.17.

   (b) Show that if u is bounded below then it must be constant.

7. Let f be analytic on the open connected set A and suppose that there is a $z_0 \in A$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in A$. Show that f is constant on A.

8.• Let f be entire and let $|f(z)| \leq M$ for z on the circle $|z| = R$; let R be fixed. Prove that

$$|f^{(k)}(re^{i\theta})| \leq \frac{k!M}{(R-r)^k} \quad k = 0, 1, 2, \ldots$$

for all $0 \leq r < R$.

9. Find a harmonic conjugate for $u(x, y) = \dfrac{x^2 + y^2 - x}{(x-1)^2 + y^2}$ on a suitable domain.

10. Let f be analytic on A and let $f'(z_0) \neq 0$. Show that if γ is a sufficiently small circle centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_\gamma \frac{dz}{f(z) - f(z_0)}.$$

Hint: Use the Inverse Function Theorem.

11. Evaluate $\displaystyle\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}} \, d\theta$.

12.• Let f and g be analytic in a region A and let $g'(z) \neq 0$ for all $z \in A$; let g be one to one and let γ be a closed curve in A. Then for z not on γ, prove that

$$f(z)I(\gamma; z) = \frac{g'(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{g(\zeta) - g(z)} d\zeta.$$

Hint: Apply the Cauchy Integral Formula to $h(\zeta) = f(\zeta)(\zeta - z)/(g(\zeta) - g(z))$ for $z \neq \zeta$ and $h(\zeta) = f(\zeta)/g'(\zeta)$. Apply this result to the case in which $g(z) = e^z$.

13. Simplify: $e^{\log i}; \log i; \log(-i); i^{\log(-1)}$.

14. Let $A = \mathbb{C}$ minus the negative real axis and zero. Show that $\log z = \int_{\gamma_z} d\zeta/\zeta$, where γ_z is any curve in A joining 1 to z. Is A simply connected?

15. Let f be analytic on a region A and let f be nonzero. Let γ be a closed curve homotopic to a point in A. Show that

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0.$$

16.• Let f be analytic on and inside the unit circle. Suppose that the image of the unit circle $|z| = 1$ lies in the disk $D = \{z \text{ such that } |z - z_0| < r\}$. Show that the image of the whole inside of the unit circle lies in D. Illustrate with e^z.

17. Is $\displaystyle\int_\gamma x\,dx + x\,dy$ always zero if γ is a closed curve?

18. Show that Poisson's Formula may be written

$$u(z) = \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\gamma_r}\frac{\zeta+z}{\zeta-z}u(\zeta)\frac{d\zeta}{\zeta}\right].$$

Use this to write a formula for the harmonic conjugate of u. Use this formula to show that if $u(\zeta)$ is continuous on the boundary, then $u(z)$ is harmonic inside it.

19. Let $f = u + iv$ be analytic on a region A. Indicate which of the following are analytic on A:

(a) $u - iv$

(b) $-u - iv$

(c) $iu - v$

20. If f is analytic on and inside the unit disk, then show that

$$f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{i\theta})}{1 - re^{i(\phi-\theta)}}d\theta \quad r < 1.$$

21. Compute the cube roots of $8i$.

22. Discuss the following sketch of a proof for Cauchy's Theorem: Suppose f is analytic in a convex region G containing 0 and γ is a closed curve in G. Define $F(t) = t\int_\gamma f(tz)dz$ for $0 \le t \le 1$. Cauchy's Theorem is that $F(1) = 0$. Compute that $F'(t) = \int_\gamma f(tz)dz + t\int_\gamma zf'(tz)dz$, and integrate the second integral by parts to obtain

$$F'(t) = \int_\gamma f(tz)dz + t\left\{\left[\frac{zf(tz)}{t}\right]_\gamma - \frac{1}{t}\int_\gamma f(tz)dz\right\} = 0$$

so that $F(1) = F(0) = 0$.[15]

23.• Prove ***Harnack's inequality***: If u is harmonic and nonnegative for $|z| \le R$, then

$$u(0)\frac{R-|z|}{R+|z|} \le u(z) \le u(0)\frac{R+|z|}{R-|z|}.$$

24. Prove the Deformation Theorem by differentiating under the integral sign and integrating by parts, assuming that these operations are valid.

[15] See P. M. Morse and H. Feshbach, *Methods of Mathematical Physics*, Part I (New York: McGraw-Hill, 1953), 364–365.