Calculus II - Midterm examination

Duration: 90 minutes.

Documents and electronic devices are forbidden. According to Nagoya University Student Discipline Rules (article 5), cheating can lead, in addition to disciplinary action, to the loss of all credits earned in all subjects during the semester.

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 40. The number of points for each problem is specified between parenthesis. Each question will be graded independently: do not hesitate to skip some of them.

Problem 1 : (9) For each of the following limits, tell if it exists. If it is the case, compute it. If it does not exist, prove it.

1. $\displaystyle\lim_{\substack{(x,y)\to(a,a)\\ x^2\neq y^2}} \frac{4x^2-3xy-y^2+2x-2y}{x^2-y^2}$. [Attention: this limit depends on $a\in\mathbb{R}$.]

2. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ (x,y)\neq\vec{0}}} \frac{3x^2-y^2}{x^2+2y^2}$. [Hint: you can consider several directions.]

Problem 2 : (8) We consider the function

$$\begin{aligned} f:\mathbb{R}^2 &\to \mathbb{R}\\ (x,y) &\mapsto x\sin(xy^2). \end{aligned}$$

1. Suppose that the graph of f admits a tangent plan at $(1/2,\sqrt{\pi}, f(1/2,\sqrt{\pi}))$. What would be its equation?

2. Prove that it is actually a tangent plan (*i.e.* f is differentiable at $(1/2,\sqrt{\pi})$).

Problem 3 : (10) We consider the function

$$\begin{aligned} f:\mathbb{R}^2 &\to \mathbb{R}^2\\ (x,y) &\mapsto \left(\frac{x^2y}{2+x^2+y^2}, \exp(x)+y^2\right) \end{aligned}$$

1. Compute the Jacobian matrix of f (= matrix of the partial derivatives) on $\mathbb{R}^2$.

2. Prove that f is differentiable on $\mathbb{R}^2$.

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Problem 4: (13) We want to study the function

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} y \sin \frac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases} \end{aligned}$$

1. Is f injective (= one-one)?
2. Is f surjective (= onto)?
3. Prove that f is continuous on $\{(x, y) \in \mathbb{R}^2 \,|\, y \neq 0\}$.
4. Prove that f is continuous on $\{(a, 0) \in \mathbb{R}^2 \,|\, a \in \mathbb{R}\}$ (the x-axis). [Hint: use the inequality $|\sin \theta| \leqslant 1$ for any $\theta \in \mathbb{R}$ and the sandwich theorem.]