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Integral Formulas for the WZNW Correlation Functions

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Abstract

We derive the integral formulas for the WZNW correlation functions, based on the Wakimoto realization. The essential idea of our derivation is to use the *"Ward identity for the screening currents"*. We also discuss the supersymmetric generalization.

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1. Introduction

In the minimal conformal models [1], the correlation functions of the chiral primary fields are given as certain integrals of hypergeometric type [2], based on the free field realization of the Virasoro algebra [3]. This type of the integral representation is generalized to other models and now it seems that any rational CFT admits such integral representation.

For the Wess-Zumino-Novikov-Witten (WZNW) models [4, 5, 6], the free field realization of the Kac-Moody algebras (Wakimoto realization) has been studied by several authors [7, 8, 9], and their general properties are clarified [10, 14, 16]. So far to derive the integral representation, straightforward application of the Wakimoto realization has not been so successful except for some particular cases [10, 11]. This is entirely due to the complicated structure of this realization [12].

The aim of the present paper is to overcome this difficulty and derive the integral representation for the WZNW models corresponding to arbitrary simple Lie algebras. Without treating the complicated explicit form of the Wakimoto realization, we can get the concrete result by using the *"Ward identity for the screening currents"*.

On the other hand, there is an independent approach to the integral representations [17, 18, 19] based on the generalized theory of hypergeometric type equations [20]. In particular, Schechtman and Varchenko proposed general integral representations for the solution of the Knizhnik-Zamolodchikov (KZ) equation [19]. We find that our result agrees with them completely.

This paper is arranged as follows. In section 2, we briefly review the Wakimoto realization. We then study the integral formulas for the WZNW correlation functions in section 3. This section contains the main results of the paper. In section 4, we discuss the supersymmetric generalization. In Appendix A, the free field realizations of the finite dimensional simple Lie algebras and the Kac-Moody algebras are summarized. In Appendix B, we describe the Schechtman-Varchenko solution.

2. The Wakimoto Realization

We start with recapitulating the results of the Wakimoto realization [10, 14, 16]. For more details we refer to Appendix A.

§ **2.1.** Let $\mathbf{g}$ be a finite dimensional semisimple Lie algebra over $\mathbf{C}$. Let e_{α_i},f_{α_i} and h_i ($i = 1, \cdots, l =$rank$\mathbf{g}$) be the Chevalley generators of $\mathbf{g}$ with the commutation relations;

$$\begin{aligned} [h_i, h_j] &= 0, & [e_{\alpha_i}, f_{\alpha_j}] &= \delta_{ij} h_j, \\ [h_i, e_{\alpha_j}] &= a_{ij} e_{\alpha_j}, & [h_i, f_{\alpha_j}] &= -a_{ij} f_{\alpha_j}. \end{aligned} \tag{2.1}$$

where α_i is the simple root, and a_{ij} is the Cartan matrix. The cartan matrix is realized as $a_{ij} = (\nu_i, \alpha_j)$, where $\nu_i = \frac{2}{\alpha_i^2}\alpha_i$ is the coroot of α_i, and (,) is the symmetric bilinear form.

The affine Kac-Moody algebra $\hat{\mathbf{g}}$ associated with the Lie algebra $\mathbf{g}$ is defined by the operator product expansions (OPE);

$$\begin{aligned} H_i(z)H_j(w) &= \frac{k}{(z-w)^2}(\nu_i, \nu_j) + \cdots, \\ H_i(z)E_{\alpha_j}(w) &= \frac{1}{z-w} a_{ij} E_{\alpha_j}(w) + \cdots, \\ H_i(z)F_{\alpha_j}(w) &= \frac{-1}{z-w} a_{ij} F_{\alpha_j}(w) + \cdots, \\ E_{\alpha_i}(z)F_{\alpha_j}(w) &= \frac{k}{(z-w)^2}\frac{2}{\alpha_i^2}\delta_{ij} + \frac{1}{z-w}\delta_{ij} H_j(w) + \cdots, \end{aligned} \tag{2.2}$$

where k is the level. The energy-momentum tensor $T_{Sug}(z)$ is given by the Sugawara construction;

$$T_{Sug}(z) = \kappa^{-1} : \sum_{i=1}^{l} H_i(z)H^i(z) + \sum_{\alpha>0} \frac{\alpha^2}{2}\big(E_\alpha(z)F_\alpha(z) + F_\alpha(z)E_\alpha(z)\big) :, \tag{2.3}$$

where $\kappa = k + h$ and h is the dual Coxeter number of $\mathbf{g}$.

For each positive root $\alpha \in \Delta_+$, we introduce bosons $\beta_\alpha(z)$ and $\gamma^\alpha(z)$, with conformal weights 1 and 0, satisfying the canonical OPE;

$$\beta_\alpha(z)\gamma^\beta(w) = \frac{1}{z-w}\delta_\alpha^\beta + \cdots, \tag{2.4}$$

The energy-momentum tensor $T_{\beta\gamma}(z)$ is

$$T_{\beta\gamma}(z) = \sum_{\alpha>0} : \partial\gamma^\alpha(z)\beta_\alpha(z) : . \tag{2.5}$$

We also introduce free bosons $\phi_i(z)$'s for $i = 1, \cdots, l$, with the OPE;[†]

$$\phi_i(z)\phi_j(w) = \kappa^{-1}(\nu_i, \nu_j) log(z-w) + \cdots, \tag{2.6}$$

from which we construct the energy-momentum tensor of Feigin-Fucks type;

$$T_\phi(z) = \sum_{i=1}^{l} : \frac{\kappa}{2}\partial\phi_i(z)\partial\phi^i(z) - \rho_i\partial^2\phi^i(z) :, \tag{2.7}$$

where $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ is the half sum of positive roots.

Then we have

2.1.1. THEOREM. [14]. *The Kac-Moody algebra* (2.2) *can be realized by the free fields* (2.4) *and* (2.6) *as follows*

$$\begin{aligned}
E_\alpha(z) &= \sum_{\beta>0} : V_\alpha^\beta\big(\gamma(z)\big)\beta_\beta(z) :, \\
H_i(z) &= -\sum_{\alpha>0}(\nu_i, \alpha) : \gamma^\alpha(z)\beta_\alpha(z) : + \kappa\partial\phi_i(z), \\
F_{\alpha_i}(z) &= \sum_{\beta>0} : V_{-\alpha_i}^\beta\big(\gamma(z)\big)\beta_\beta(z) : + \kappa\partial\phi_i(z)\gamma(z)^{\alpha_i} + c_i\partial\gamma^{\alpha_i}(z),
\end{aligned} \tag{2.8}$$

$$T_{Sug}(z) = T_{\beta\gamma}(z) + T_\phi(z),$$

where $V_\alpha^\beta(x)$ *and* $V_{-\alpha_i}^\beta(x)$ *are the polynomials of* x^α *defined in Appendix A. The*

† Our notation for $\phi(z)$ is different from ordinary one. To get the ordinary $\phi(z)$, we must substitute $\phi(z) \to i\,\kappa^{-1/2}\phi(z)$.

coefficient c_i *in* $F_{\alpha_i}(z)$ *is given by* [13]

$$c_i = \frac{2}{{\alpha_i}^2}k + \frac{h - {\alpha_i}^2}{{\alpha_i}^2}. \tag{2.9}$$

§ **2.2.** There exists one more important ingredient in the Wakimoto realization, that is the "screening current".

Define the "screening current" $s_i(z)$, corresponding to the simple root α_i ;

$$\begin{aligned} s_i(z) &= S_{\alpha_i}(z) e^{-\alpha_i \cdot \phi(z)}, \\ S_\alpha(z) &= \sum_{\beta > 0} : S_\alpha^\beta\big(\gamma(z)\big) \beta_\beta(z) :, \end{aligned} \tag{2.10}$$

where $S_\alpha^\beta(x)$ is also a polynomial of x^α given in Appendix A.

Then we get

2.2.1. PROPOSITION. *The "screening current"* $s_i(z)$ *satisfies*

$$\begin{aligned} E_\alpha(z) s_j(w) &= 0 + \cdots, \\ H_i(z) s_j(w) &= 0 + \cdots, \\ F_{\alpha_i}(z) s_j(w) &= -\kappa \delta_{ij} \frac{2}{{\alpha_i}^2} \frac{\partial}{\partial w} \big(\frac{1}{z-w} e^{-\alpha_j \cdot \phi(w)} \big) + \cdots, \\ T_{Sug}(z) s_j(w) &= \frac{\partial}{\partial w} \big(\frac{1}{z-w} s_j(w) \big) + \cdots. \end{aligned} \tag{2.11}$$

Note that the screening charge $\int dt e^{-\alpha \cdot \phi(t)} S_\alpha(t)$ is well defined only for the simple roots α_i's, though $S_\alpha(t)$ is well-defined for all the positive roots.

§ **2.3.** Now we give a natural highest weight representation. For arbitrary weight vector λ, the vertex operator $e^{\lambda \cdot \phi(z)}$ satisfies the highest weight condition;

$$\begin{aligned} E_\alpha(z) e^{\lambda \cdot \phi(w)} &= 0 + \cdots, \\ H_i(z) e^{\lambda \cdot \phi(w)} &= \frac{\lambda_i}{z-w} e^{\lambda \cdot \phi(w)} + \cdots. \end{aligned} \tag{2.12}$$

It has the conformal weight $(\lambda, \lambda + 2\rho)/\kappa$ w.r.t. the $T_{sug}(z)$.

For "ordered" set of simple roots, $I = \{\alpha_1, \cdots, \alpha_r\}$,† the fields

$$e^{\lambda\cdot\phi(z)} P^I_\lambda(z) \equiv \int \prod_{i=1}^{r} dt_i F_{\alpha_i}(t_i) e^{\lambda\cdot\phi(z)} \tag{2.13}$$

form the basis of the descendants of the highest weight vector $e^{\lambda\cdot\phi(z)}$.‡ Since the $\partial\gamma(z)$ terms (2.8) give no singularity, they don't contribute to this integral. So the operator $P^I_\lambda(z) \in \mathbf{C}[\gamma^\alpha(z)]$ can be constructed from a classical polynomial $P^I_\lambda \in \mathbf{C}[x^\alpha]$ below (A.10) by replacing x^α with $\gamma^\alpha(z)$.

Hence the vectors $e^{\lambda\cdot\phi(z)} P_\lambda(z)$, with arbitrary polynomial $P_\lambda(z) \in \mathbf{C}[\gamma^\alpha(z)]$, form the highest weight representation of the Wakimoto realization.

3. Integral Formulas from the Wakimoto Realization

We prove the Schechtman-Varchenko integral formulas [19] for the solution of the KZ equation by using the Wakimoto realization.

§ **3.1.** In the free field realization, a correlation function of the chiral primary fields is represented by the correlation of the screening charges as well as the vertex operators. For WZNW models it takes the form;

$$\int \prod_{i=1}^{m} dt_i \langle \prod_{i=1}^{m} e^{-\alpha_i\cdot\phi(t_i)} S_i(t_i) \prod_{a=1}^{n} e^{\lambda_a\cdot\phi(z_a)} P_a(z_a)\rangle. \tag{3.1}$$

Here $\alpha_1, ..., \alpha_m$ are simple roots that fill the gap between the highest weights of the incoming states $\lambda_1, ..., \lambda_n$ and the outgoing state λ_∞, i.e.

$$\sum_{a=1}^{n} \lambda_a - \lambda_\infty = \sum_{i=1}^{m} \alpha_i. \tag{3.2}$$

In the set $\{\alpha_1, ..., \alpha_m\}$ or $\{\lambda_1, ..., \lambda_n\}$ some α's or λ's may be repeated.

† We sometimes use over-simplified notation $I = \{\alpha_1, \cdots, \alpha_r\}$ to denote an ordered set I of simple roots α_i, although they should be understood as $\{\alpha_{i_1}, \cdots, \alpha_{i_r}\}$.

‡ In what follows the operator factors in product symbol $\prod$ are assumed to be ordered in the sense that $\prod_{i\in\{1,2,4\}} \mathbf{O}_i = \mathbf{O}_1\mathbf{O}_2\mathbf{O}_4 \quad (\neq \mathbf{O}_2\mathbf{O}_1\mathbf{O}_4)$.

Calculation of the ϕ field correlation is standard and the result is given by the difference products;

$$
\begin{aligned}
Q &\equiv \langle \prod_{i=1}^{m} e^{-\alpha_i\cdot\phi(t_i)} \prod_{a=1}^{n} e^{\lambda_a\cdot\phi(z_a)} \rangle \\
&= \prod_{i<j}^{m} (t_i - t_j)^{\frac{\alpha_i\cdot\alpha_j}{\kappa}} \prod_{i=1}^{m}\prod_{a=1}^{n} (t_i - z_a)^{\frac{-\alpha_i\cdot\lambda_a}{\kappa}} \prod_{a<b}^{n} (z_a - z_b)^{\frac{\lambda_a\cdot\lambda_b}{\kappa}}.
\end{aligned}
\tag{3.3}
$$

§ **3.2.** Let us next calculate the β-γ correlation

$$
\omega = \langle \prod_{i=1}^{m} S_{\alpha_i}(t_i) \prod_{a=1}^{n} P_a(z_a) \rangle. \tag{3.4}
$$

This form ω is single valued meromorphic function (1-form) of z's (t's).

Direct evaluation of ω is quite cumbersome since the explicit form of $S_\alpha(t)$ and $P(z)$ is complicated in general. We find, however, that ω can be calculated without using the explicit formula for $S_\alpha(t)$ and $P(z)$. All we need is the following OPE relations,

3.2.1. LEMMA. *The $\beta\gamma$ parts of the screening currents and the vertex operator satisfy*

$$
\begin{aligned}
S_\alpha(t_1) S_\beta(t_2) &= \sum_{\gamma>0} \frac{1}{t_1 - t_2} f_{\alpha\beta}^{\gamma} S_\gamma(t_2) + \cdots, \\
S_\alpha(t) P(z) &= \frac{1}{t - z} (S_\alpha P)(z) + \cdots,
\end{aligned}
\tag{3.5}
$$

where $f_{\alpha\beta}^{\gamma}$ are the structure constants of $\mathbf{n}_+$ and the operator $(S_\alpha P)(z) \in \mathbf{C}[\gamma^\alpha(z)]$ corresponds to the polynomial $(S_\alpha P) = \sum_\beta S_\alpha^\beta(x) \frac{\partial}{\partial x^\beta} P(x) \in \mathbf{C}[x^\alpha]$.

This lemma follows form the fact that $S_\alpha(t)$ is constructed in terms of "left $\mathbf{n}_+$- current", and $S_\alpha(z)$ and $P(z)$ are of order 1 and 0 w.r.t. $\beta_\alpha(z)$'s, respectively.

Using these relations, we obtain the *"Ward identity for the screening currents"*;

$$
\begin{aligned}
&\langle S_\alpha(t) S_{\alpha_1}(t_1)\cdots S_{\alpha_m}(t_m) P_1(z_1)\cdots P_n(z_n)\rangle \\
&\quad = \sum_{i=1}^{m} \frac{1}{t-t_i} f^{\beta}_{\alpha\alpha_i} \langle S_{\alpha_1}(t_1)\cdots S_\beta(t_i)\cdots S_{\alpha_m}(t_m) P_1(z_1)\cdots P_n(z_n)\rangle \\
&\quad + \sum_{a=1}^{n} \frac{1}{t-z_a} \langle S_{\alpha_1}(t_1)\cdots S_{\alpha_m}(t_m) P_1(z_1)\cdots S_\alpha P_a(z_a)\cdots P_n(z_n)\rangle .
\end{aligned}
\tag{3.6}
$$

Iterative use of this equation gives

3.2.2. PROPOSITION. *The $\beta\gamma$ correlation ω is given by*

$$
\omega = \sum_{part} \prod_{a=1}^{n} \langle \prod_{i\in I_a} S_{\alpha_i}(t_i) P_a(z_a)\rangle , \tag{3.7}
$$

$$
\begin{aligned}
&\langle S_{\alpha_1}(t_1)\cdots S_{\alpha_q}(t_q) P(z)\rangle \\
&\quad = \sum_{perm} \frac{1}{(t_1-t_2)(t_2-t_3)\cdots(t_q-z)} \langle (S_{\alpha_1}\cdots S_{\alpha_q} P)(z)\rangle ,
\end{aligned}
\tag{3.8}
$$

where $\sum_{part}$ stands for the summation over all the partition of $I=\{1,2,\cdots,m\}$ into n disjoint union $I_1\cup I_2\cup\cdots\cup I_n$ and $\sum_{perm}$ the summation over all the permutation of the elements of $\{1,2,\cdots,q\}$.

The proof is given by induction, or checking the OPE directly. In this proof, it is used that the $\beta\gamma$ propagator $\frac{1}{z-w}$ satisfies; $\frac{1}{t_1-t_2}\cdot\frac{1}{t_2-t_3} = \frac{1}{t_1-t_3}\cdot\{\frac{1}{t_2-t_3}-\frac{1}{t_2-t_1}\}$.

Note that $\langle S_{\alpha_1}(t_1)\cdots S_{\alpha_q}(t_q)P(z)\rangle$ is symmetric with respect to $S_{\alpha_i}(t_i)$'s, but $\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle$ depends on the ordering of S_{α_i}'s (see §3.3 below). We see proposition 3.2.2 is similar to the 1st solution (B.4), (B.5) presented by Schechtman and Varchenko [19].

3.2.3. EXAMPLE. The first few examples are

$$
\begin{aligned}
\langle S_\alpha(t)P(z)\rangle &= \frac{\langle (S_\alpha P)(z)\rangle}{t-z}, \\
\langle S_{\alpha_1}(t_1)S_{\alpha_2}(t_2)P(z)\rangle &= \frac{\langle (S_{\alpha_1}S_{\alpha_2}P)(z)\rangle}{(t_1-t_2)(t_2-z)} + \frac{\langle (S_{\alpha_2}S_{\alpha_1}P)(z)\rangle}{(t_2-t_1)(t_1-z)}, \\
\langle S_{\alpha_1}(t_1)S_{\alpha_2}(t_2)P_1(z_1)P_2(z_2)\rangle &= \langle S_{\alpha_1}(t_1)S_{\alpha_2}(t_2)P_1(z_1)\rangle\langle P_2(z_2)\rangle \\
&\quad + \langle S_{\alpha_1}(t_1)P_1(z_1)\rangle\langle S_{\alpha_2}(t_2)P_2(z_2)\rangle \\
&\quad + \langle S_{\alpha_2}(t_2)P_1(z_1)\rangle\langle S_{\alpha_1}(t_1)P_2(z_2)\rangle \\
&\quad + \langle P_1(z_1)\rangle\langle S_{\alpha_1}(t_1)S_{\alpha_2}(t_2)P_2(z_2)\rangle.
\end{aligned} \tag{3.9}
$$

§ **3.3.** Our final task is to evaluate $\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle$. We will show that this correlation can be calculated by using only finite dimensional algebra.

Note that only the c-number term of the operator $(S_{\alpha_1}\cdots S_{\alpha_q}P)(z) \in \mathbf{C}[\gamma^\alpha(z)]$ has nonzero contribution to the correlation $\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle$. Moreover the constant term of the operator $(S_{\alpha_1}\cdots S_{\alpha_q}P)(z) \in \mathbf{C}[\gamma^\alpha(z)]$ is the same as that of the polynomial $S_{\alpha_1}\cdots S_{\alpha_q}P \in \mathbf{C}[x^\alpha]$. Therefore we can evaluate this correlator by using the differential operators defined in Appendix A, as;

$$
\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle = \langle 0|S_{\alpha_1}\cdots S_{\alpha_q}P|0\rangle, \tag{3.10}
$$

where $\frac{\partial}{\partial x^\alpha}|0\rangle = 0$, $\langle 0|x^\alpha = 0$ and $\langle 0|0\rangle = 1$.

Since $\langle 0|S_\alpha = -\langle 0|E_\alpha$ and $[S_\alpha, E_\beta] = 0$, we have

$$
\langle 0|S_{\alpha_1}\cdots S_{\alpha_q}P|0\rangle = -\langle 0|S_{\alpha_2}\cdots S_{\alpha_q}E_{\alpha_1}P|0\rangle. \tag{3.11}
$$

Thus we have proved

3.3.1. LEMMA. *The correlator $\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle$ is nothing but the "Shapovalov form" up to sign,*

$$
\langle (S_{\alpha_1}\cdots S_{\alpha_q}P)(z)\rangle = (-1)^q\langle 0|E_{\alpha_q}\cdots E_{\alpha_1}P|0\rangle, \tag{3.12}
$$

where $E_\alpha|0\rangle = 0$, $H_i|0\rangle = \lambda_i|0\rangle$, $\langle 0|H_i = \lambda_i\langle 0|$, $\langle 0|F_\alpha = 0$ and $\langle 0|0\rangle = 1$.

For the basis vector $P^I|0\rangle = \prod_{j=1}^r F_{\beta_j}|0\rangle$, the Shapovalov form enjoys the inductive formula;

$$\langle 0|\prod_{i=1}^q S_{\alpha_i}\prod_{j=1}^r F_{\beta_j}|0\rangle = \sum_{k=1}^q \delta_{\beta_1}^{\alpha_k}\frac{2}{\alpha_k^2}(\sum_{l=k+1}^q \alpha_k\cdot\alpha_l - \alpha_k\cdot\lambda)\langle 0|\prod_{i\neq k}^q S_{\alpha_i}\prod_{j=2}^r F_{\beta_j}|0\rangle. \quad (3.13)$$

Combining this Shapovalov form and proposition 3.2.2 , we get

$$\begin{aligned}&\langle\prod_{i=1}^q S_{\alpha_i}(t_i)P_\lambda^{\{\beta_1,\cdots,\beta_r\}}(z)\rangle\\ &\quad = \sum_{k=1}^q \delta_{\beta_1}^{\alpha_k}\frac{2}{\alpha_k^2}(\sum_{l\neq k}^q \frac{\alpha_k\cdot\alpha_l}{t_k-t_l} - \frac{\alpha_k\cdot\lambda}{t_k-z})\langle\prod_{i\neq k}^q S_{\alpha_i}(t_i)P_\lambda^{\{\beta_2,\cdots,\beta_r\}}(z)\rangle.\end{aligned} \quad (3.14)$$

Iterative use of this equation gives the expression for ω. Most of the terms in (3.8) vanish unless $\sum_{i=1}^q \alpha_i = \sum_{j=1}^r \beta_j$, i.e. $\{\alpha_1,\cdots,\alpha_q\} = \{\beta_1,\cdots,\beta_r\}$ as a "set". Then we have

3.3.2. PROPOSITION. *For the basis vectors $P_{\lambda_a}^{I_a}(z_a)$, the $\beta\gamma$ correlation ω is given by*

$$\omega = \sigma\prod_{a=1}^n\langle\prod_{i\in I_a} S_{\alpha_i}(t_i)P_{\lambda_a}^{I_a}(z_a)\rangle, \quad (3.15)$$

$$\langle\prod_{i=1}^r S_{\alpha_i}(t_i)P_\lambda^{\{\alpha_1,\cdots,\alpha_r\}}(z)\rangle = \sigma\prod_{k=1}^r\{\frac{2}{\alpha_k^2}(\sum_{l=k+1}^r \frac{\alpha_k\cdot\alpha_l}{t_k-t_l} - \frac{\alpha_k\cdot\lambda}{t_k-z})\}, \quad (3.16)$$

where σ is the symmetrization of the variable t's associated with the same α's.

The proof is given by induction. Note that $\langle\prod_{i=1}^r S_{\alpha_i}(t_i)P_\lambda^{\{\alpha_1,\cdots,\alpha_r\}}(z)\rangle$ depends on the ordering of α_i's of $P_\lambda^{\{\alpha_1,\cdots,\alpha_r\}}$, although it is symmetric w.r.t. $S_{\alpha_i}(t_i)$'s. We see that proposition 3.3.2 coincides with the 2nd solution of Schechtman and Varchenko (B.6), (B.7) up to an overall factor [19].

3.3.3. EXAMPLE. The first few examples are

$$\langle 0|S_\alpha F_\beta|0\rangle = \delta^\alpha_\beta \frac{2}{\alpha^2}(-\alpha\cdot\lambda),$$
$$\langle 0|S_{\alpha_1}S_{\alpha_2}F_{\beta_1}F_{\beta_2}|0\rangle = \frac{2}{\alpha_1^2}\frac{2}{\alpha_2^2}\{\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}(\alpha_1\cdot\alpha_2-\alpha_1\cdot\lambda)(-\alpha_2\cdot\lambda)$$
$$+\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_1}_{\beta_2}(-\alpha_2\cdot\lambda)(-\alpha_1\cdot\lambda)\},$$

$$\langle S_\alpha(t)P^{\{\beta\}}_\lambda(z)\rangle = \delta^\alpha_\beta\frac{2}{\alpha^2}\Big(-\frac{\alpha\cdot\lambda}{t-z}\Big),$$
$$\langle S_{\alpha_1}(t_1)S_{\alpha_2}(t_2)P^{\{\beta_1,\beta_2\}}_\lambda(z)\rangle = \frac{2}{\alpha_1^2}\frac{2}{\alpha_2^2}\{\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\Big(\frac{\alpha_1\cdot\alpha_2}{t_1-t_2}-\frac{\alpha_1\cdot\lambda}{t_1-z}\Big)\Big(-\frac{\alpha_2\cdot\lambda}{t_2-z}\Big)$$
$$+\delta^{\alpha_2}_{\beta_1}\delta^{\alpha_1}_{\beta_2}\Big(\frac{\alpha_2\cdot\alpha_1}{t_2-t_1}-\frac{\alpha_2\cdot\lambda}{t_2-z}\Big)\Big(-\frac{\alpha_1\cdot\lambda}{t_1-z}\Big)\}. \tag{3.17}$$

§ **3.4.** Finally we have the main theorem

3.4.1. THEOREM. *The formulas* (3.3), (3.7) *or* (3.15) *give the integral representation of the solution of KZ eq.* $(B.1)$ *in the form*

$$\int\prod_{i=1}^m dt_i\, Q(t_1,\cdots,t_m;z_1,\cdots,z_n)\ \omega(t_1,\cdots,t_m;z_1,\cdots,z_n). \tag{3.18}$$

For more complete expressions see Appendix B.

4. Generalization to the super WZNW models

We now turn to the supersymmetric generalization of the Wakimoto realization and the integral formulas for the correlation functions [27].

§ **4.1.** The Wakimoto realization of the super Kac-Moody algebras are essentially given by the same way as the non super case.

The affine super Kac-Moody algebra is defined by

$$
\begin{aligned}
H_i(z_1,\theta_1)H_j(z_2,\theta_2) &= \frac{k}{z_1-z_2-\theta_1\theta_2}(\nu_i,\nu_j)+\cdots,\\
H_i(z_1,\theta_1)E_{\alpha_j}(z_2,\theta_2) &= \frac{\theta_1-\theta_2}{z_1-z_2}a_{ij}E_{\alpha_j}(z_2,\theta_2)+\cdots,\\
H_i(z_1,\theta_1)F_{\alpha_j}(z_2,\theta_2) &= -\frac{\theta_1-\theta_2}{z_1-z_2}a_{ij}F_{\alpha_j}(z_2,\theta_2)+\cdots,\\
E_{\alpha_i}(z_1,\theta_1)F_{\alpha_j}(z_2,\theta_2) &= \frac{k}{z_1-z_2-\theta_1\theta_2}\frac{2}{\alpha_i^2}\delta_{ij}+\frac{\theta_1-\theta_2}{z_1-z_2}\delta_{ij}H_j(z_2,\theta_2)+\cdots,
\end{aligned}
\tag{4.1}
$$

where θ is a grassmann coordinate with conformal dimension $-\frac{1}{2}$.

Corresponding to the free fields $\beta_\alpha(z)$, $\gamma^\alpha(z)$ and $\phi_i(z)$'s, we introduce bosons $B_\alpha(z,\theta)$ and $\Gamma^\alpha(z,\theta)$ (spin $\frac{1}{2},0$) and free bosons $\Phi_i(z,\theta)$'s, such that

$$
\begin{aligned}
B_\alpha(z_1,\theta_1)\Gamma^\beta(z_2,\theta_2) &= \frac{\theta_1-\theta_2}{z_1-z_2}\delta_\alpha^\beta+\cdots,\\
\Phi_i(z_1,\theta_1)\Phi_j(z_2,\theta_2) &= \kappa^{-1}(\nu_i,\nu_j)log(z_1-z_2-\theta_1\theta_2)+\cdots.
\end{aligned}
\tag{4.2}
$$

Notice that this $B\Gamma$ propagator is nilpotent.

Then we have

4.1.1. PROPOSITION. *The super Kac-Moody algebra* (4.1) *can be realized in terms of the free fields* (4.2)*. It is given from proposition* 2.1.1 *by replacing* $\beta_\alpha(z)$*,* $\gamma^\alpha(z)$ *and* $\phi_i(z)$ *with their corresponding super fields* $B_\alpha(z,\theta)$*,* $\Gamma^\alpha(z,\theta)$ *and* $\Phi_i(z,\theta)$*, and* ∂ *with the covariant derivative* $D=\frac{\partial}{\partial\theta}+\theta\frac{\partial}{\partial z}$*, and by changing the coefficients as ;* $\kappa=k$ *and* $c_i=\frac{2}{{\alpha_i}^2}k$*.*

Because of the nilpotency of the propagator, there is no anomaly in the OPE , $:V_\alpha^\beta\big(\Gamma(z_1,\theta_1)\big)B_\beta(z_1,\theta_1)::V_\gamma^\delta\big(\Gamma(z_2,\theta_2)\big)B_\delta(z_2,\theta_2):$. Therefore the coefficients κ and c_i have rather simple form than non super cases.

We introduce the screening current $s_i(z,\theta)$ and $S_\alpha(z,\theta)$ defined by the similar way to (2.10), with the same polynomial $S_\alpha^\beta(x)$.

4.1.2. PROPOSITION. *The screening current $s_i(z,\theta)$ satisfies*

$$\begin{aligned} E_\alpha(z_1,\theta_1)s_j(z_2,\theta_2) &= 0+\cdots, \\ H_i(z_1,\theta_1)s_j(z_2,\theta_2) &= 0+\cdots, \\ F_{\alpha_i}(z_1,\theta_1)s_j(z_2,\theta_2) &= \kappa\delta_{ij}\frac{2}{{\alpha_i}^2}D_2\big(\frac{\theta_1-\theta_2}{z_1-z_2}e^{-\alpha_j\cdot\Phi(z_2,\theta_2)}\big)+\cdots. \end{aligned} \tag{4.3}$$

The vertex operator $e^{\lambda\cdot\Phi(z,\theta)}P_\lambda(z,\theta)$ is also given by the similar way to the non super case e.g. (2.13), with the same polynomial $P(x)$.

§ **4.2.** We next give the integral formulas for the super WZNW correlation functions. The results are almost the same as the last chapter's except for the form of propagators.

A correlation function of the chiral primary fields is given as similar to (3.1), with the super fields $\Phi(z,\theta)$, $S(z,\theta)$ and $P(z,\theta)$, and an extra integral $\int\prod_{i=1}^m d\theta_i$. We obtain the Φ field correlation Q by replacing (t_i-t_j) with $(t_i-t_j-\theta_i\theta_j)$, etc. in (3.3).

Calculation of the $B\Gamma$ correlation ω is analogous to the last chapter's. The reason is as follows;
(1). $S_\alpha(t,\theta)$ and $P(z,\theta)$ satisfy the similar OPE in Lemma 3.2.1, with the propagator $\frac{\theta_1-\theta_2}{t_1-t_2}$, since they are constructed in the same way as the non super case.
(2). The $B\Gamma$ propagator satisfies the relation; $\frac{\theta_1-\theta_2}{t_1-t_2}\cdot\frac{\theta_2-\theta_3}{t_2-t_3}=\frac{\theta_1-\theta_3}{t_1-t_3}\cdot\{\frac{\theta_2-\theta_3}{t_2-t_3}-\frac{\theta_2-\theta_1}{t_2-t_1}\}$.
(3). The Shapovalov form does not change, since it has nothing to do with supersymmetrization.

But ω is antisymmetric under the exchange of $S_\alpha(t,\theta)$'s, so there appears an extra phase factor $(-1)^\sigma$ under these permutation. Hence we obtain

4.2.1. PROPOSITION. *The $B\Gamma$ correlation ω is calculated from the proposition* 3.2.2 *and* 3.3.2*, with the propagator $\frac{\theta_1-\theta_2}{t_1-t_2}$ and the extra phase factor $(-1)^\sigma$ w.r.t. the permutation of $\sum_{part}$, $\sum_{perm}$ and the symmetrization σ .*

Finally we have

4.2.2. PROPOSITION. *Our formulas for the super WZNW correlation functions give the integral representation of the solution of super KZ eq.* $(B.3)$.

5. Conclusion

Using the Wakimoto realization and the *"Ward identity for the screening currents"* (3.6), we have derived the correlation function of the chiral primary field of the WZNW models on P^1 explicitly. The results coincide with the formula presented by Schechtman and Varchenko. Thus we have presented the natural and simple field theoretical proof of the Schechtman-Varchenko integral formulas.

The integral representation provides an efficient tool to investigate the monodromy properties of the conformal blocks and their relation to the quantum groups [4, 16, 23, 24]. We hope our formulation gives further insight into this problem.

Appendix A

We briefly summarize the free field realization of the finite dimensional simple Lie algebras and the Kac-Moody algebras.

§ **A.1.** For a finite dimensional simple Lie group G, the free field realization is given by the Borel-Weil-Bott theorem. In the following we fix the Gauss decomposition of $G = N_- \cdot H \cdot N_+$ and that of their Lie algebra $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$.

Let x^α $(\alpha \in \Delta_+)$ be a coordinate of the flag manifolds such that $z \equiv \exp(\sum_\alpha x^\alpha e_\alpha) \in Y$, where $Y = B_- \backslash G$, $B_- = N_- \cdot H$ is the Borel subgroup. The algebra $\mathbf{g}$ is realized on the polynomial ring $\mathbf{C}[x^\alpha]$ as (twisted first order) differential operators induced by "right action" G on Y. The differential operators E_α, F_α and H_i corresponding to the Chevalley generators e_α, f_α and h_i are given by

$$E_\alpha = \sum_{\beta>0} V_\alpha^\beta(x)\frac{\partial}{\partial x^\beta},$$

$$H_i = \sum_{\beta>0} V_i^\beta(x)\frac{\partial}{\partial x^\beta} + \lambda_i, \tag{A.1}$$

$$F_{\alpha_i} = \sum_{\beta>0} V_{-\alpha_i}^\beta(x)\frac{\partial}{\partial x^\beta} + \lambda_i x^{\alpha_i},$$

where λ is the highest weight of the representation and $V(x)$'s are defined by

$$z\cdot \exp(te_\alpha) = \exp\big(t\sum_{\beta>0} V_\alpha^\beta(x)\frac{\partial}{\partial x^\beta} + O(t^2)\big)\cdot z,$$

$$z\cdot \exp(th_i) = \exp(th_i)\ \exp\big(t\sum_{\beta>0} V_i^\beta(x)\frac{\partial}{\partial x^\beta} + O(t^2)\big)\cdot z,$$

$$z\cdot \exp(tf_{\alpha_i}) = \exp(tf_{\alpha_i})\ \exp(th_i x^{\alpha_i})\ \exp\big(t\sum_{\beta>0} V_{-\alpha_i}^\beta(x)\frac{\partial}{\partial x^\beta} + O(t^2)\big)\cdot z, \tag{A.2}$$

and

$$V_\alpha^\beta(x) = \delta_\alpha^\beta - \frac{1}{2}\sum_{\gamma>0} f_{\alpha\gamma}^\beta x^\gamma + O(x^2),$$

$$V_i^\beta(x) = -(\nu_i,\beta)x^\beta, \tag{A.3}$$

$$V_{-\alpha_i}^\beta(x) = -\sum_{\gamma>0} f_{-\alpha_i\gamma}^\beta x^\gamma - \frac{1}{2}(\nu_i,\beta)x^{\alpha_i}x^\beta + O(x^3).$$

There is another type of differential operators S_α induced by "left action" of $e_\alpha \in \mathbf{n}_+$ that plays an important role in this paper. They are given by

$$S_\alpha = \sum_{\beta>0} S_\alpha^\beta(x)\frac{\partial}{\partial x^\beta},$$

$$\exp(-te_\alpha)\cdot z = \exp\big(t\sum_{\beta>0} S_\alpha^\beta(x)\frac{\partial}{\partial x^\beta} + O(t^2)\big)\cdot z, \tag{A.4}$$

and

$$S_\alpha^\beta(x) = -\delta_\alpha^\beta - \frac{1}{2}\sum_{\gamma>0} f_{\alpha\gamma}^\beta x^\gamma + O(x^2). \tag{A.5}$$

By using the associativity of the algebra, we have

$$\begin{aligned} [E_\alpha, S_\beta] &= 0, \\ [H_i, S_\alpha] &= (\nu_i, \alpha) S_\alpha, \\ [F_{\alpha_i}, S_{\alpha_j}] &= -(\nu_i, \alpha_j)\, x^{\alpha_i} S_{\alpha_j} - \lambda_i \delta_{ij}. \end{aligned} \tag{A.6}$$

§ **A.2.** The free field realization of the Kac-Moody currents $E_\alpha(z)$, $F_\alpha(z)$, $H_i(z)$ and screening currents $S_\alpha(z)$ is essentially obtained from E_α, F_α, H_i and S_α by the substitution [16];

$$\frac{\partial}{\partial x^\alpha} \to \beta_\alpha(z), \qquad x^\alpha \to \gamma^\alpha(z), \qquad \lambda_i \to \kappa \partial \phi_i(z). \tag{A.7}$$

Here we should take into account the corrections due to the normal ordering effect.

For example, the OPE of two operators of the form $V_\alpha(z) = \sum_{\beta>0} : V_\alpha^\beta(\gamma(z))\beta_\beta(z) :$ corresponding to $V_\alpha = \sum_{\beta>0} V_\alpha^\beta(x) \frac{\partial}{\partial x^\beta}$ are given by

$$\begin{aligned} V_\alpha(z) V_\beta(w) = &- \frac{1}{(z-w)^2} \sum_{\mu,\nu>0} \partial_\mu V_\alpha^\nu \partial_\nu V_\beta^\mu(\gamma(w)) \\ &+ \frac{1}{z-w} \sum_{\mu,\nu>0} : \left(V_\alpha^\nu \partial_\nu V_\beta^\mu - V_\beta^\nu \partial_\nu V_\alpha^\mu\right)(\gamma(w)) \beta_\mu(w) : \\ &+ \frac{1}{z-w} \sum_{\mu,\nu,\rho>0} \partial\gamma^\rho(w) \partial_\rho \partial_\mu V_\alpha^\nu \partial_\nu V_\beta^\mu(\gamma(w)) + \cdots, \end{aligned} \tag{A.8}$$

where $\partial = \frac{\partial}{\partial w}$ and $\partial_\alpha V_\beta^\gamma(\gamma(w)) \in \mathbf{C}[\gamma(w)]$ corresponds to $\frac{\partial}{\partial x^\alpha} V_\beta^\gamma(x) \in \mathbf{C}[x]$. In general, the coefficients of the $\frac{1}{(z-w)^2}$ terms are polynomials of $\gamma(w)$'s, and they are different from the required form (2.2). The coefficient vanishes if it has a positive weight w.r.t. H. So, by counting the weight, it turns out that the dangerous OPE's are FE, FH, FF and HH.

As is discussed in [14], these anomalous terms can be adjusted to the form of (2.2), by the renormalization of $k : \kappa = k + h$ and adding $\partial\gamma^\alpha(z)$ terms to $F_\alpha(z)$, as follows†

$$\begin{aligned} F_\alpha(z) = \sum_{\beta>0} : V^\beta_{-\alpha}\big(\gamma(z)\big)\beta_\beta(z) : + \kappa \sum_{i=1}^{rank\mathbf{g}} \partial\phi_i(z) W^i_{-\alpha}\big(\gamma(z)\big) \\ + \sum_{\beta>0} \Lambda_{-\alpha,\beta}\big(\gamma(z)\big)\partial\gamma^\beta(z), \end{aligned} \tag{A.9}$$

where $W^i_{-\alpha}\big(\gamma(z)\big), \Lambda_{-\alpha,\beta}\big(\gamma(z)\big) \in \mathbf{C}[\gamma(z)]$, and h is the dual Coxeter number of $\mathbf{g}$. For simple root α_i, (A.9) is given explicitly as the proposition 2.1.1. The other $F_\alpha(z)$ are constructed from this $F_{\alpha_i}(z)$ through OPE.

§ **A.3.** Now we give the representation. For a simple Lie algebra $\mathbf{g}$, the Verma module V_λ is generated by the highest weight vector $|\lambda\rangle$ which satisfies $e_\alpha|\lambda\rangle = 0$, $h_i|\lambda\rangle = \lambda_i|\lambda\rangle$. Dual module V^*_λ is generated by $\langle\lambda|$ which satisfies $\langle\lambda|f_\alpha = 0$, $\langle\lambda|h_i = \lambda_i\langle\lambda|$. The bilinear form $V^*_\lambda \otimes V_\lambda \to \mathbf{C}$ defined by $\langle\lambda|\lambda\rangle = 1$ is called Shapovalov form.

For the free field realization of the simple Lie algebras, the state $|0\rangle \equiv 1 \in \mathbf{C}[x^\alpha]$ is the highest weight vector such that

$$\frac{\partial}{\partial x^\alpha}|0\rangle = 0, \qquad E_\alpha|0\rangle = 0, \qquad H_i|0\rangle = \lambda_i|0\rangle. \tag{A.10}$$

The vectors $P^I_\lambda|0\rangle = \prod_{\alpha_i\in I} F_{\alpha_i}|0\rangle \in \mathbf{C}[x^\alpha]$, for ordered set $I = \{\alpha_1, \cdots, \alpha_r\}$, form the basis of the descendants of $|0\rangle$.

We also introduce $\langle 0| \in Hom(\mathbf{C}[x^\alpha], \mathbf{C})$ that maps any polynomial $P(x^\alpha)$ to its constant term $P(0)$, then we have

$$\langle 0|x^\alpha = 0, \qquad \langle 0|F_\alpha = 0, \qquad \langle 0|H_i = \lambda_i\langle 0|. \tag{A.11}$$

The $\langle 0|$, $|0\rangle$ expectation value gives the Shapovalov form.

† A complete proof of this statement is given in [15] by using Lie algebra cohomology.

For the Wakimoto realization of the Kac-Moody algebras, the representation is given as §2.3.‡

Our notation is summarized as follows

	abstract algebra	*$B_-\backslash G$ realization*	*Wakimoto realization*
generators	$e_\alpha,\ f_\alpha,\ h_i$	$E_\alpha,\ F_\alpha,\ H_i$	$E_\alpha(z),\ F_\alpha(z),\ H_i(z)$
module	V_λ	$\mathbf{C}[x^\alpha]$	$\mathbf{C}[\gamma^\alpha(z)]e^{\lambda\cdot\phi(z)}$
h.w.v.	$\vert\lambda\rangle$	$\vert 0\rangle$	$e^{\lambda\cdot\phi(z)}$

Appendix B

In this Appendix we summarize the solution of the KZ equation proposed by Schechtman and Varchenko.

§ **B.1.** The KZ equation is formulated as follows. Fix the highest weights of the incoming states and outgoing states as (3.2). Put $L \equiv V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. The vectors $f_I|\lambda\rangle \equiv \prod_{a=1}^n \prod_{i\in I_a} f_{\alpha_i}|\lambda_a\rangle$, for ordered partition of $\{1, 2, \cdots, m\}$ into n disjoint union $I = I_1 \cup I_2 \cup \cdots \cup I_n$, form the basis of L. Similarly $\langle\lambda|e_I$ forms the basis of L^*. The Shapovalov form can be generalized on $L^* \otimes L$.

In WZNW model the correlation function of the chiral primary fields satisfies the KZ equation [5];

$$\kappa\frac{\partial}{\partial z_a}|\Phi(z_1,\cdots,z_n)\rangle = \sum_{b(\neq a)} \frac{\Omega_{ab}}{z_a - z_b}|\Phi(z_1,\cdots,z_n)\rangle, \tag{B.1}$$

where $|\Phi(z)\rangle$ takes the value in L and

$$\Omega_{ab} = \sum_{i=1}^{dim\mathbf{g}} 1 \otimes \cdots \otimes \overset{(a)}{X}_i \otimes \cdots \otimes \overset{(b)}{X}_i \otimes \cdots \otimes 1 \tag{B.2}$$

with orthonormal basis $X_i \in \mathbf{g}$.

‡ The structure of this representation was analyzed in [14, 16, 22] by similar method in [21].

In super WZNW model the super KZ equation is given by [25, 26]

$$\kappa\Big(\frac{\partial}{\partial\theta_a}+\theta_a\frac{\partial}{\partial z_a}\Big)|\Phi(z_1,\theta_1,\cdots,z_n,\theta_n)\rangle=\sum_{b(\neq a)}\frac{\theta_a-\theta_b}{z_a-z_b}\Omega_{ab}|\Phi(z_1,\theta_1,\cdots,z_n,\theta_n)\rangle. \tag{B.3}$$

§ **B.2.** The first solution of Schechtman and Varchenko takes the form of L-valued integral;

$$|\Phi(z)\rangle=\int\prod_{i=1}^{m}dt_iQ\sum_{part}\prod_{a=1}^{n}\prod_{i\in I_a}f_{\alpha_i}(t_i)|\psi_{\lambda_a}(z_a)\rangle, \tag{B.4}$$

where Q is given in (3.3) and

$$\prod_{i=1}^{q}f_{\alpha_i}(t_i)|\psi_\lambda(z)\rangle\equiv\sum_{perm}\frac{1}{(t_1-t_2)(t_2-t_3)\cdots(t_q-z)}\,f_{\alpha_1}\cdots f_{\alpha_q}|\lambda\rangle. \tag{B.5}$$

We can show that $|\Phi(z)\rangle$ is a highest weight vector w.r.t. diagonal $\Delta(\mathbf{g})$ with highest weight λ_∞. This solution corresponds to the result of proposition 3.2.2.

The 2nd solution is the contravariant form of the 1st one w.r.t. the Shapovalov form $L^*\otimes L\to\mathbf{C}$. Its value on $\langle\lambda|e_I\in L^*$ is written as

$$\langle\lambda|\,e_I\,|\Phi(z)\rangle=\int\prod_{i=1}^{m}dt_iQ\sigma\prod_{a=1}^{n}\langle\prod_{i\in I_a}G_{\alpha_i}(t_i)\rangle_{\lambda_a}(z_a), \tag{B.6}$$

where σ is the symmetrization of the variable t's associated with the same α's and

$$\begin{aligned}\langle\,1\,\rangle_\lambda(z)&=\,1,\\ \langle G_\alpha(t)\prod_{i=1}^{n}G_{\alpha_i}(t_i)\rangle_\lambda(z)&=\Big(\sum_{i=1}^{n}\frac{\alpha\cdot\alpha_i}{t-t_i}-\frac{\alpha\cdot\lambda}{t-z}\Big)\langle\prod_{i=1}^{n}G_{\alpha_i}(t_i)\rangle_\lambda(z).\end{aligned} \tag{B.7}$$

Note that the meromorphic function $\langle\prod_{i=1}^{r}G_{\alpha_i}(t_i)\rangle_\lambda(z)$ depends on the ordering of the G's. This solution agrees with the result of proposition 3.3.2 up to an overall factor.

These forms also enjoy the following interesting identities;

$$\sigma\langle\prod_{i=1}^{n} G_\alpha(t_i)\rangle_\lambda(z) = \prod_{j=0}^{n-1}(\,j\frac{\alpha\cdot\alpha}{2} - \alpha\cdot\lambda)\prod_{j=1}^{n}\frac{1}{t_j - z}, \tag{B.8}$$

and

$$\begin{aligned} &\sigma\sum_{r=0}^{n}(-1)^r\binom{n}{r}\ \langle\prod_{i=1}^{r} G_\alpha(t_i)\,G_\beta(y)\prod_{i=r+1}^{n} G_\alpha(t_i)\prod_{i=1}^{m} G_{\alpha_i}(x_i)\rangle_\lambda(z) \\ &= \prod_{j=0}^{n-1}(\,j\frac{\alpha\cdot\alpha}{2} + \alpha\cdot\beta)\prod_{j=1}^{n}\frac{1}{t_j - y}\{\sum_{i=1}^{n}\phi_\alpha(t_i) + \phi_\beta(y)\}\langle\prod_{i=1}^{m} G_{\alpha_i}(x_i)\rangle_\lambda(z), \end{aligned} \tag{B.9}$$

where ϕ is given by

$$\phi_\alpha(t) = \sum_{i=1}^{m}\frac{\alpha\cdot\alpha_i}{t - x_i} - \frac{\alpha\cdot\lambda}{t - z}. \tag{B.10}$$

These formulas yield the following relations;

$$\begin{aligned} \langle {G_\alpha}^{n+1}\rangle_\lambda = 0, &\qquad n = 2\,\frac{\alpha\cdot\lambda}{\alpha\cdot\alpha}, \\ \{Ad\ G_\alpha\}^{n+1}\cdot G_\beta = 0, &\qquad n = -2\,\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}, \end{aligned} \tag{B.11}$$

which can be understood from (3.5) naturally.

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