Fusion Rules for the Fractional Level $\widehat{sl}(2)$ Algebra

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Abstract

We derive, as the condition for the null vector decoupling, the fusion rules for the $\widehat{sl}(2)$ algebra with fractional level, which have an interesting structure related to affine Weyl transformation. It is shown that, to get nontrivial fusion rules, we must include the primary field which belongs to neither the highest nor the lowest weight representations.
1. Introduction

In case of the integral representation, i.e. the level $k$ is a non negative integer, the fusion rules of the $\hat{sl}(2)$ affine Lie algebra are well known [1, 2, 3]. The generalization for other levels, however, is not straightforward and still remains as an open problem.

For example, in the admissible representations [4, 5], which are deeply related to Virasoro minimal models, the fusion rules calculated from the Verlinde formula [6] give strange results; they contain negative integers [5, 7]. Furthermore, Tsuchiya and Wakimoto [8] showed, by an analysis similar to that in [1], that almost all the primary fields vanish.

In the present paper, we investigate this problem with a slightly modified definition of the primary fields, and give some meaningful results. This paper is arranged as follows. In section 2, we briefly summarize the structure of the modules over the $\hat{sl}(2)$ algebra. In section 3, we introduce the modified primary field and calculate its matrix elements including the null vector. We give the fusion rules for the generic level in section 4, and for the admissible representations in section 5. Finally, we discuss the operator product expansion in section 6.

2. $\hat{sl}(2)$ Algebra

§2.1. First we define some notation. The $\hat{sl}(2)$ algebra is generated by $E_n$, $H_n$ and $F_n$ ($n \in \mathbb{Z}$) with

\begin{align*}
[H_n, E_m] &= E_{n+m}, & [H_n, H_m] &= \frac{k}{2} n \delta_{n+m,0}, \\
[H_n, F_m] &= -F_{n+m}, & [E_n, F_m] &= 2H_{n+m} + kn \delta_{n+m,0}.
\end{align*}

Let $M_j$ be the Verma module over $\hat{sl}(2)$, generated by the highest weight vector $|j\rangle$, such that $E_n|j\rangle = H_n|j\rangle = F_n|j\rangle = 0$ ($n > 0$), $E_0|j\rangle = 0$ and $H_0|j\rangle = j|j\rangle$. 

This \( |j \rangle \) is primary with respect to the Sugawara energy-momentum tensor

\[
L_n = \frac{1}{k+2} \sum_{m \in \mathbb{Z}} :2H_m H_{n-m} + F_m E_{n-m} + E_m F_{n-m} : \quad (2.2)
\]

with \( c_k = 3k/(k+2) \), \( h_j = j(j+1)/(k+2) \). Note that \( k, j \in \mathbb{C} \).

The dual module \( M_j^* \) is generated by \( \langle j | \) which satisfies \( \langle j | E_n = \langle j | H_n = \langle j | F_n = 0 \) \( (n < 0) \), \( \langle j | F_0 = 0 \) and \( \langle j | H_0 = j \langle j | \). The bilinear form \( M_j^* \otimes M_j \rightarrow \mathbb{C} \) is uniquely defined by \( \langle j | j \rangle = 1 \) and \( \langle (u | X) | v \rangle = \langle u | (X | v) \rangle \) for any \( \langle u | \in M_j^* \), \( | v \rangle \in M_j \) and \( X \in \widehat{sl(2)} \).

\[2.2.\] A null vector \( | \chi \rangle \in M_j \) (of grade \( N \) and charge \( Q \)) is defined by \( E_n | \chi \rangle = H_n | \chi \rangle = F_n | \chi \rangle = 0 \) \((n > 0)\), \( E_0 | \chi \rangle = 0 \), \( H_0 | \chi \rangle = (j + Q) | \chi \rangle \) and \( L_0 | \chi \rangle = (h_j + N) | \chi \rangle \). A null vector \( \langle \chi | \in M_j^* \) is defined in a similar manner.

Let \( t = k + 2 \), then we have the following \([9, 10]\),

**THEOREM I.** For \( t \in \mathbb{C} \setminus \{0\} \) and the highest weight \( j \), parametrized as \( 2j_{r,s} + 1 = r - st \) with \( r, s \in \mathbb{Z} \), such that (I) \( r > 0 \) and \( s \geq 0 \) or (II) \( r < 0 \) and \( s < 0 \), there exist a unique null vector \( | \chi_{r,s} \rangle \in M_j \) of grade \( N = rs \) and charge \( Q = -r \).

And the null vector in \( M_{j_{r,s}} \) is as following

For (I),

\[
| \chi_{r,s} \rangle = (F_0)^{r+st}(E_{-1})^{r+(s-1)t} \cdots (E_{-1})^{r-(s-1)t}(F_0)^{r-st}|j_{rs} \rangle. \quad (2.3)
\]

For (II),

\[
| \chi_{r,s} \rangle = (E_{-1})^{-r-(s+1)t}(F_0)^{-r-(s+2)t} \cdots (F_0)^{-r+(s+2)t}(E_{-1})^{-r+(s+1)t}|j_{rs} \rangle. \quad (2.4)
\]

Note that the exponents of these formulas are complex numbers, but they make sense by analytic continuation. More explicit formulas for the null vectors are given in \([12]\). Some examples and the relation to Virasoro null vectors are given in \([13]\).
Let the proper maximal submodule of $M_j$ be $SM_j$, then the quotient $L_j = M_j / SM_j$ gives the irreducible module. The $L_j$ fall into three cases

1. If $2j + 1 \neq r - st$, with the above $r$ and $s$, (I) (II), then $L_j = M_j$.
2. If $2j + 1 = r - st$ and $t$ is not a rational number, then $SM_j$ is generated by single null vector $|\chi_{r,s}\rangle$.
3. If $2j + 1 = r - st$ and $t$ is a nonzero rational number $p/q$, then $SM_j$ is generated by two null vectors.

3. Primary Field

§ 3.1. For the triple $(j_3 j_2 j_1)$, the primary field $\phi_{j_2}(z, x) : M^*_{j_3} \otimes M_{j_1} \rightarrow \mathbb{C}$ is defined by [3, 12]

\[
[F_n, \phi_j(z, x)] = z^n \frac{d}{dx} \phi_j(z, x)
\]
\[
[H_n, \phi_j(z, x)] = z^n (-x \frac{d}{dx} + j) \phi_j(z, x)
\]
\[
[E_n, \phi_j(z, x)] = z^n (-x^2 \frac{d}{dx} + 2xj) \phi_j(z, x)
\]
and
\[
[L_n, \phi_j(z, x)] = z^n (z \frac{d}{dz} + h_j(n + 1)) \phi_j(z, x).
\]

If $\phi_j(z, x)$ is polynomial in $x$, then this is equivalent to the ordinary definition [1], in case of fractional level, however, we have to allow non-polynomial dependence in $x$. Introducing $J_n(y) = E_n - 2yH_n - y^2F_n$, (3.1) takes the following compact form,

\[
[J_n(y), \phi_j(z, x)] = z^n (- (x - y)^2 \frac{d}{dx} + 2j(x - y)) \phi_j(z, x).
\]

From the conservation of the $H_0$ and $L_0$ charges, $\langle j_3 | H_0 \phi_{j_2}(z, x) | j_1 \rangle = \langle j_3 | [H_0, \phi_{j_2}(z, x)] | j_1 \rangle + \langle j_3 | \phi_{j_2}(z, x) H_0 | j_1 \rangle$, (similarly for $L_0$); the ground state
matrix element of the primary field is given by

$$\langle j_3 | \phi_{j_2}(z, x) | j_1 \rangle = C_{123} z^{h_1-h_2+h_3} x^{j_1+j_2-j_3}, \quad (3.3)$$

where $C_{123}$ is an arbitrary constant. The other matrix elements for the descendant fields is uniquely determined by $sl(2)$ invariance

$$\langle u | X \phi_{j_2}(z, x) | v \rangle = \langle u | [X, \phi_{j_2}(z, x)] | v \rangle + \langle u | \phi_{j_2}(z, x) X | v \rangle, \quad (3.4)$$

for any $\langle u | \in M_{j_3}^+, | v \rangle \in M_{j_1}$ and $X \in \widehat{sl}(2)$. Note that, if $C_{123} = 0$, then $\phi_{j_2}(z, x) = 0$.

§ 3.2. We now consider the matrix element including the null vector

$$\langle j_3 | \phi_{j_2}(z, x) | \chi_{r,s} \rangle = C_{123} f_{r,s}(j_1, j_2, j_3) z^{h_1-h_2+h_3-rs} x^{j_1+j_2-j_3-r} \quad (3.5)$$

From the Theorem I, (3.1) and (3.3), we can calculate the function $f_{r,s}(j_1, j_2, j_3)$

**PROPOSITION II.**

For (I),

$$f_{r,s}(j_1, j_2, j_3) = \prod_{n=0}^{r-1} \prod_{m=0}^{s} (j_1+j_2-j_3-nmt) \prod_{n=1}^{r} \prod_{m=1}^{s} (-j_1+j_2+j_3+nmt). \quad (3.6)$$

For (II),

$$f_{r,s}(j_1, j_2, j_3) = \prod_{n=0}^{-r-1} \prod_{m=0}^{-s-1} (-j_1+j_2+j_3+nmt) \prod_{n=1}^{-r} \prod_{m=1}^{-s-1} (j_1+j_2-j_3+nmt). \quad (3.7)$$

**Proof.** For any $n, j \in \mathbb{C}$, we have

$$\partial^n x^j = \frac{j!}{(j-n)!} x^{j-n}$$

$$(-x^2 \partial + 2xj_2)^n x^j = \frac{(2j_2-j)!}{(2j_2-j-n)!} x^{j+n}, \quad (3.8)$$

with the gamma function $n! = \Gamma(n+1)$. Using the Theorem I, (3.1) and (3.3),
we need only to calculate (with $j = j_1 + j_2 - j_3$)

\[
\partial^{r+st}(-x^2\partial + 2xj_2)^{r+(s-1)t} \cdots (-x^2\partial + 2xj_2)^{r-(s-1)t}\partial^{r-st}x^j
\]

\[
= \frac{(j+st)! (2j_2 - j - t + r)! \cdots (2j_2 - j - st + r)! (j)!}{(j-r)! (2j_2 - j - st)! \cdots (2j_2 - j - t)! (j+st-r)!}
\]

\[
= \frac{(j+st)! (2j_2 - j - t + r)! \cdots (2j_2 - j - st + r)! (j)!}{(j+st-r)! (2j_2 - j - t)! \cdots (2j_2 - j - st)! (j-r)!}x^{j-r}.
\]

This gives the finite product (3.6) for integers $r, s \in \mathbb{Z}$. Similarly,

\[
(-x^2\partial + 2xj_2)^{-r-(s+1)t}\partial^{-r-(s+2)t} \cdots \partial^{-r+(s+2)t}(-x^2\partial + 2xj_2)^{-r+(s+1)t}x^j
\]

\[
= \frac{(2j_2 - j - (s+1)t)! (j - t - r)! \cdots (j + (s+1)t)! (2j_2 - j)!}{(2j_2 - j - (s+1)t + r)! (j - t)! \cdots (j + (s+1)t + r)! (2j_2 - j + r)!}x^{j-r}
\]

(3.10)

gives (3.7).

Q.E.D.

§ 3.3. Another matrix element $\langle \chi_{r,s}\phi_{j_2}(z, x)|j_1 \rangle$ follows from $\langle j_3|\phi_{j_2}(z, x)|\chi_{r,s} \rangle$ by using the anti-algebra automorphism $\sigma$ such that, $\sigma(E_n) = -F_{-n}$, $\sigma(H_n) = H_{-n}$, $\sigma(F_n) = -E_{-n}$, $\sigma(dz^hdx^{-j}\phi_j(z, x)) = dw^hdy^{-j}\phi_j(w, y)$ and $\sigma(dz^hdx^{-j}\langle u|\phi_j(z, x)|v \rangle) = dz^hdx^{-j}\langle u|\phi_j(z, x)|v \rangle$, with $w = z^{-1}, y = x^{-1}$. We obtain

\[
\langle \chi_{r,s}\phi_{j_2}(z, x)|j_1 \rangle = C_{123}f_{r,s}(j_3, j_2, j_1)z^{-h_1-h_2+h_3+r}x^{j_1+j_2-j_3+r},
\]

(3.11)

with the same function $f_{r,s}(j_3, j_2, j_1)$ as (3.5).
4. Fusion Rules for the Generic Level

§ 4.1. When $M_{j_1}$ has a null vector $|\chi_{r,s}\rangle$, there exists a primary field $\phi_{j_2}(z, x) : M_{j_3}^* \otimes L_{j_1} \rightarrow \mathbb{C}$ for the irreducible module $L_{j_1}$, if and only if the matrix element including the null vector vanish;

$$\langle j_3 | \phi_{j_2}(z, x) | \chi_{r,s}\rangle = 0. \tag{4.1}$$

For the nonzero constant $C_{123}$, the null state decoupling condition (4.1) is equivalent to $f_{r,s}(j_1, j_2, j_3) = 0$. And this gives the fusion rules.

Let us introduce the Weyl transformations, $\sigma_0 : j \mapsto -j - 1$ and $\sigma_1 : j \mapsto t - j - 1$, then we have

**THEOREM III.** In the generic level $t \in \mathbb{C}$, the fusion rules for $j_1 = j_{r,s}$ and arbitrary $j_2$, are as follows

For (I), the allowed value of $j_3$ is

$$j_3 = \sigma(-j_1 + j_2 + n), \tag{4.2}$$

where $0 \leq n \leq r - 1$, $n \in \mathbb{Z}$, and $\sigma = \sigma_1, \sigma_0 \sigma_1, \sigma_1 \sigma_0 \sigma_1, \ldots, (\sigma_0 \sigma_1)^{2s}$.

For (II), the allowed value of $j_3$ is

$$j_3 = \sigma(j_1 - j_2 + n), \tag{4.3}$$

where $0 \leq n \leq -r - 1$, $n \in \mathbb{Z}$, and $\sigma = \sigma_1, \sigma_0 \sigma_1, \sigma_1 \sigma_0 \sigma_1, \ldots, (\sigma_0 \sigma_1)^{-2(s+1)}$.

**Proof.** From the Proposition II, and using $2j_1 + 1 = r - st$, we have for (I), the allowed value of $j_3$ is either of the following:

\begin{align*}
(a) \quad & j_3 = -j_1 + j_2 + n - mt \\
(b) \quad & j_3 = j_1 - j_2 - 1 + t - n' + m't
\end{align*}

\tag{4.4}

where $0 \leq n \leq r - 1$, $0 \leq m \leq s$, $0 \leq n' \leq r - 1$ and $0 \leq m' \leq s - 1$. 

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For (II), the allowed value of $j_3$ is either of the following:

$$(c) \quad j_3 = j_1 - j_2 + u - wt$$

$$(d) \quad j_3 = -j_1 + j_2 - 1 + t - u' + w't$$

where $0 \leq u \leq -r - 1$, $0 \leq w \leq -s - 1$, $0 \leq u' \leq -r - 1$ and $0 \leq w' \leq -s - 2$.

The above solutions are related by the Weyl transformations as follows, $\sigma_1: (a) \to (b)$, with $n' = n$ and $m' = m$ for $m \neq s$; $\sigma_0: (b) \to (a)$, with $n = n'$ and $m = m' + 1$; $\sigma_1: (c) \to (d)$, with $u' = u$ and $w' = w$ for $w \neq -s - 1$; and $\sigma_0: (d) \to (c)$, with $u = u'$ and $w = w' + 1$. Q.E.D.

We can arrange the allowed values of $j_3$ as follows

\[
\begin{align*}
J_1^1 & \iff J_2^1 \iff J_3^1 \iff \cdots \iff J_{2s+1}^1 \iff J_{2s+1}^1 \\
\downarrow & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\
J_1^2 & \iff J_2^2 \iff J_3^2 \iff \cdots \iff J_{2s+1}^2 \iff J_{2s+1}^2 \\
\downarrow & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
J_1^{[r]} & \iff J_2^{[r]} \iff J_3^{[r]} \iff \cdots \iff J_{2s+1}^{[r]} \iff J_{2s+1}^{[r]} \\
\downarrow & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \cdots & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow & \uparrow \downarrow
\end{align*}
\]

for (I), $J_1^1 = -j_1 + j_2$ and $J_{2s+1}^{[r]} = j_1 + j_2$;

for (II), $J_1^1 = j_1 - j_2$ and $J_{2s+1}^{[r]} = -j_1 - j_2 + t - 2$;

where $J \leftrightarrow J'$, $J \leftrightarrow J'$ and $J \to J'$ show that $J' = \sigma_0 J = -J - 1$, $J' = \sigma_1 J = t - J - 1$ and $J' = J + 1$ respectively.

§ 4.2. In the case that $M_{j_3}^*$ also has a null vector $\langle \chi_{r,s} |$, the existence condition for the primary field $\phi_{j_2}(z, x) : L_{j_3}^* \otimes L_{j_1} \to \mathbf{C}$, is not only (4.1) but also $\langle \chi_{r,s} | \phi_{j_2}(z, x) |j_1\rangle = 0$. And this null state decoupling condition is equivalent to $f_{r_1,s_1}(j_1, j_2, j_3) = f_{r_3,s_3}(j_3, j_2, j_1) = 0$. 

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5. Fusion Rules for the Admissible Representations

§ 5.1. If the level is rational \( t = p/q \), with the coprime integers \( p \) and \( q \), then \( j_{r,s} = j_{r-p,s-q} \). Hence, there are two independent null vectors for \( j_1 = j_{r,s} = j_{r-p,s-q} \). There exists a primary field \( \phi_{j_2}(z,x) : M^*_j \otimes L_j \to \mathbb{C} \) if and only if

\[
\langle j_3|\phi_{j_2}(z,x)|\chi_{r,s}\rangle = \langle j_3|\phi_{j_2}(z,x)|\chi_{r-p,s-q}\rangle = 0.
\] (5.1)

For the nonzero constant \( C_{123} \), the null state decoupling condition (5.1) is equivalent to the condition Cond\((j_1, j_2, j_3, r, s)\)

\[
\prod_{n=0}^{r-1} \prod_{m=0}^{s} (j_1 + j_2 - j_3 - n + mt) \prod_{n=1}^{r} \prod_{m=1}^{s} (-j_1 + j_2 + j_3 + n - mt) \prod_{n=0}^{r-p} \prod_{m=0}^{q-s} (-j_1 + j_2 + j_3 - n + mt) \prod_{n=1}^{r} \prod_{m=1}^{s} (j_1 + j_2 - j_3 + n - mt) = 0.
\] (5.2)

Then we have

THEOREM IV. For the rational level \( t = p/q \), with the coprime integers \( p \) and \( q \), the fusion rules are closed in the grid \( 1 \leq r_i \leq p - 1 \) and \( 0 \leq s_i \leq q - 1 \) for \( j_i = r_i - s_i t \). And they are given by

\[
\max(-r_1 + r_2 + 1, r_1 - r_2 + 1) \leq r_3 \leq \min(r_1 + r_2 - 1, -r_1 - r_2 - 1 + 2p)
\]

\[
\max(-s_1 + s_2, s_1 - s_2) \leq s_3 \leq \min(s_1 + s_2, -s_1 - s_2 - 2 + 2q)
\]

or

\[
\max(-r_1 - r_2 + 1 + p, r_1 + r_2 + 1 - p) \leq r_3 \leq \min(r_1 - r_2 - 1 + p, -r_1 + r_2 - 1 + p)
\]

\[
\max(-s_1 - s_2 + q, s_1 + s_2 + 2 - q) \leq s_3 \leq \min(s_1 - s_2 - 2 + q, -s_1 + s_2 - 2 + q)
\] (5.3)

where \( r_3 \) and \( s_3 \) runs through even (or odd) integers if they are bounded by even (or odd) integers respectively.
Proof. From (5.2), the fusion rule is
\[
\begin{align*}
2j_2 + 1 &= (n + n' + 1) - (m + m')t \\
2j_3 + 1 &= (r + n - n') - (s + m - m')t,
\end{align*}
\] (5.4)
where \(0 \leq n \leq r - 1, \ 0 \leq m \leq s, \ 0 \leq n' \leq p - r - 1 \) and \(0 \leq m' \leq q - s - 1\), or
\[
\begin{align*}
2j_2 + 1 &= (p - 1 - n - n') - (q - 2 - m - m')t \\
2j_3 + 1 &= (r - n + n') - (s - m + m')t,
\end{align*}
\] (5.5)
where \(0 \leq n \leq r - 1, \ 0 \leq m \leq s - 1, \ 0 \leq n' \leq p - 1 - r \) and \(0 \leq m' \leq q - 2 - s\).
In general, if
\[
\begin{align*}
x &= n + m + c_x & a_n \leq n \leq b_n \\
y &= n - m + c_y & a_m \leq m \leq b_m,
\end{align*}
\] (5.6)
then
\[
\begin{align*}
-x + c_x + c_y + 2a_n &\leq y \leq -x + c_x + c_y + 2b_n \\
x - c_x + c_y - 2b_m &\leq y \leq x - c_x + c_y - 2a_m.
\end{align*}
\] (5.7)
Consequently we can get the range of the \(r, s\). Moreover, these \(j_2\) and \(j_3\) are in the same grid as \(j_1\).

Q.E.D.

§ 5.2. Now we give some examples

EXAMPLES.

(1). For \(t = k + 2, \ k \in \mathbb{Z}_{\geq 0}\); the allowed value of \(j_3\) is
\[
\max(-j_1 + j_2, j_1 - j_2) \leq j_3 \leq \min(j_1 + j_2, -j_1 - j_2 + (t - 2)),
\] (5.8)
where \(j_3\) runs from the lower bound to the upper bound by increments of 1.

(2). For \(t = \frac{2}{2n+1}, \ n \in \mathbb{Z}_{+}\); the allowed value of \(j_3\) is
\[
\max(j_1 + j_2, -j_1 - j_2 + t - 2) \leq j_3 \leq \min(-j_1 + j_2, j_1 - j_2),
\]
or
\[
\max(j_1 - j_2 + t - 1, -j_1 + j_2 + t - 1) \leq j_3 \leq \min(-j_1 - j_2 - 1, j_1 + j_2 - t + 1),
\] (5.9)
where \(j_3\) runs from the lower bound to the upper bound by increments of \(t\).
For $t = \frac{3}{2}$, the matrices $N(r,s)$, defined by

$$[r_1, s_1] \times [r_2, s_2] = \sum_{(r_3, s_3)} N_{(r_1, s_1)}^{(r_3, s_3)} [r_3, s_3],$$

(5.10)

and

$$N(r,s) = \begin{bmatrix} N_{(1,0)}(r,s) & N_{(2,0)}(r,s) & N_{(1,1)}(r,s) & N_{(2,1)}(r,s) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

(5.11)

are

$$N_{(1,0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad N_{(2,0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(5.12)

$$N_{(1,1)} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad N_{(2,1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The Verlinde formula gives rise to the strange result that $N_{(1,1)}$ and $N_{(2,1)}$ contain the negative integers ( $N_{(1,0)}$ and $N_{(2,0)}$ are the same as (5.12) )

$$N_{(1,1)} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad N_{(2,1)} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(5.13)

We will give the relation between our formula and the Verlinde’s one in section 7.
§ 5.3. The fusion rule from (5.1) says that \( M_{j_3} \) also has two null vectors. Hence, the existence condition for the primary field \( \phi_{j_2}(z, x) : L_{j_3}^* \otimes L_{j_1} \to \mathbb{C} \), is not only (5.1) but also

\[ \langle \chi_{r,s} | \phi_{j_2}(z, x) | j_1 \rangle = \langle \chi_{r-p,s-q} | \phi_{j_2}(z, x) | j_1 \rangle = 0. \]  \hspace{1cm} (5.14)

However, the condition Cond\((j_3, j_2, j_1, r, s)\) which is equivalent to (5.14), is nothing but Cond\((j_1, j_2, j_3, p - r, q - s - 1)\). Consequently, there is no change for the fusion rule (5.3).

6. The Relation to the Operator Product Expansion

The following discussion will clarify the meaning of the null vector decoupling condition. The fusion rules can be defined as the condition for the existence of the non-vanishing operator product expansion,

\[ \phi_{j_2}(z, x)|j_1\rangle = \sum_{N,Q} \sum_{J} \beta^{NQ}_J z^{h_{12} + N} x^{j_{12} + Q} |J^{NQ}\rangle_{j_3}, \]  \hspace{1cm} (6.1)

where \( h_{12} = -h_1 - h_2 + h_3, j_{12} = j_1 + j_2 - j_3 \) and \( |J^{NQ}\rangle_{j_3} \) is the basis of the homogeneous components of \( L_0 = h_3 + N, H_0 = j_3 - Q \). In this expression, the primary field \( \phi_{j_2}(z, x) \) is considered as an operator \( \phi : M_{j_1} \to M_{j_3} \) rather than the matrix element \( \phi : M_{j_3}^* \otimes M_{j_1} \to \mathbb{C} \). This reinterpretation is needed to define the product of primary fields.

To determine the coefficients \( \beta_J = \beta^{NQ}_J \) for fixed \( N, Q \) in (6.1), we must solve the following linear equation,

\[ z^{h_{12} + N} x^{j_{12} + Q} \sum_J \langle I | J \rangle \beta_J = \langle I | \phi_{j_2}(z, x) | j_1 \rangle. \]  \hspace{1cm} (6.2)

If the determinant of the matrix \( \langle I | J \rangle \) (the Shapovalov form) vanishes, the consistency of the above equation (6.2) requires some constraints.
Generally, for a linear equation on \( x, Sx = y \), the existence and uniqueness conditions of the solution are

i). \( x \) exists if and only if \( z \cdot y = 0 \) for any \( z \in \text{Coker} \, S \cong \{ z \mid zS = 0 \} \),

ii). \( x \) is unique up to \( \text{Ker} \, S = \{ x \mid Sx = 0 \} \).

In our problem (6.2), the condition corresponding to i) is the OPE existence condition in [11, 12]. Note that the determinant vanishes if and only if there exists some vector \( \langle \chi \rangle = \sum I \, C_I \langle I \rangle \) such that, \( \langle \chi \mid J \rangle = 0 \) for all \( J \)'s. Such a \( \langle \chi \rangle \) is a null vector or its descendant. Hence the OPE existence condition takes the form

\[
\langle \chi \mid \phi_{j_2}(z, x) \mid j_1 \rangle = 0,
\]

(6.3)

for any \( \langle \chi \rangle \in SM_{j_3}^\ast \). This is equivalent to the null vector decoupling condition.

Therefore, we obtain

**Proposition V.** The null vector decoupling condition is equivalent to the OPE existence condition.

According to the above condition ii), if the triple \((j_3, j_2, j_1)\) satisfies the null vector decoupling condition, then the right hand side of (6.1) exists uniquely up to the null vectors.

**7. Conclusions and Remarks**

As the condition for the null vector decoupling, we have derived the fusion rules for the \( \widehat{sl}(2) \) algebra with fractional level. The results show an interesting structure related to affine Weyl transformation. Similar results are expected for other affine Lie algebras.

We should like to add some further remarks.

(1). We found that when bra and ket spaces are admissible representations, \( L_{j_3}^\ast \otimes L_{j_1} \), the allowed value \( j_2 \) of the primary field \( \phi_{j_2} \) is also given by the weights of an admissible representation. However, it should be remarked that
the representation of \( \hat{sl}(2) \) defined over this primary field is not the \( L_{j_2} \). Indeed, it is neither the highest nor the lowest weight representation with respect to the zero mode \( sl(2) \) subalgebra, unless \( j_1 + j_2 - j_3 \equiv 0 \) or \( 2j_2 \pmod{Z} \).

Note that, for the allowed weights in Theorem IV; if \( t \in \mathbb{Z} \), then \( j_1 + j_2 - j_3 = \frac{1}{2}(r_1 + r_2 - r_3 - 1) - \frac{1}{2}(s_1 + s_2 - s_3)t \in \mathbb{Z} \); and for \( t \notin \mathbb{Z} \),

i). \( j_1 + j_2 - j_3 = n \in \mathbb{Z} \), if and only if \( s_3 = s_1 + s_2 \) and \( r_3 = r_1 + r_2 - 1 - 2n \).

ii). \( j_1 - j_2 - j_3 = -n \in \mathbb{Z} \), if and only if \( s_3 = s_1 - s_2 \) and \( r_3 = r_1 - r_2 + 1 + 2n \).

Hence almost all the allowed representations in theorem IV are not the \( L_{j_2} \).

Therefore, our results are consistent with those of Tsuchiya and Wakimoto, mentioned in the Introduction, which say that almost all the couplings of \( L^*_{j_3} \otimes L_{j_2} \otimes L_{j_1} \) vanish.

(2). The correlation function of the primary fields (3.1) satisfies the following KZ equations [3]

\[
(k + 2) \frac{\partial}{\partial z_a} - \sum_{b(\neq a)} \frac{\Omega_{ab}}{z_a - z_b}) \Psi(z_1, x_1, \cdots, z_N, x_N) = 0
\]

and has the global \( sl(2) \) invariance

\[
\sum_{a=1}^{N} x_a^n (x_a \frac{\partial}{\partial x_a} - j_a(n + 1)) \Psi(z_1, x_1, \cdots, z_N, x_N) = 0
\]

\[
\sum_{a=1}^{N} z_a^n (z_a \frac{\partial}{\partial z_a} + h_a(n + 1)) \Psi(z_1, x_1, \cdots, z_N, x_N) = 0,
\]

where \( n = -1, 0, 1 \).

In terms of the correlation function, our matrix (3.3) simply corresponds to the following three point function

\[
\Psi(z_1, x_1, z_2, x_2, z_3, x_3) = C_{123} \prod_{a<b} (z_a - z_b)^{h_{ab}} (x_a - x_b)^{j_{ab}},
\]

which is symmetric with respect to 1, 2 and 3, where \( h_{12} = -h_1 - h_2 + h_3 \) and
\[ j_{12} = j_1 + j_2 - j_3 \text{ etc.}\] Our matrix element is given by the limit \( x_3 = \infty, x_2 = x \) and \( x_1 = 0 \) (and similarly for the \( z \)’s). By this procedure, the above symmetry is broken. The representation for bra (ket) becomes the lowest (highest) weight. For the representation over the primary field, however we can say only that the positive mode operators vanish. From this point of view, the unwanted behavior of the representation over the primary field is an artifact of the operator interpretation.

(3). The dimension of the solutions of the KZ equation (7.1) is dictated by the fusion rules. We have the integral representation of the solution [14], but they are polynomials in \( x \) in the formulation presented here. To obtain all the solutions, including non-polynomial solutions, it seems that we need another type of screening charges from that considered in [15]. Indeed, in recent papers [16], Furlan et. al. construct explicitly the \( n \)-point function for the \( sl(2) \) WZW theory for non-integer level and spins. In case of fractional level \( k \), their solution involves infinite dimensional representation of \( sl(2) \).

(4). Finally we give the relation between our formula and the Verlinde’s one. The Verlinde formula corresponds to the case that the primary field \( \phi_{j_2}(z, x) \) is restricted to the highest or lowest weight representation. However, the lowest weight representation should be replaced in the Verlinde formula by \( \sigma(j_2) = -j_2 - 1 \) with a negative sign fusion coefficient.

The rule for the transformation from our formula to the Verlinde’s one is as follows:

i). If the primary field \( \phi_{j_2}(z, x) \) is the highest weight representation, i.e. \( j_1 + j_2 - j_3 \in \mathbb{Z}_{\geq 0} \), then such a fusion is allowed in the Verlinde formula.

ii). If the primary field \( \phi_{j_2}(z, x) \) is not the highest but the lowest weight representation, i.e. \( j_1 + j_2 - j_3 \notin \mathbb{Z}_{\geq 0} \) and \( j_3 + j_2 - j_1 \in \mathbb{Z}_{\geq 0} \), then the fusion with \( \sigma(j_2) = -j_2 - 1 \) is allowed with negative sign.

iii). If the primary field \( \phi_{j_2}(z, x) \) is neither the highest nor the lowest weight representation, then such a fusion is not allowed in the Verlinde formula.
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References

8. A. Tsuchiya and M. Wakimoto, private communication.

