Dihedral $G$-Hilb via representations of the McKay quiver

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Abstract

For a given finite small binary dihedral group $G \subset \text{GL}(2, \mathbb{C})$ we provide an explicit description of the minimal resolution $Y$ of the singularity $\mathbb{C}^2/G$. The minimal resolution $Y$ is known to be either the moduli space of $G$-clusters $G\text{-Hilb}(\mathbb{C}^2)$, or the equivalent $\mathcal{M}_\theta(Q, R)$, the moduli space of $\theta$-stable quiver representations of the McKay quiver. We use both moduli approaches to give an explicit open cover of $Y$, by assigning to every distinguished $G$-graph an open set $U_\Gamma \subset \mathcal{M}_\theta(Q, R)$, and calculating the explicit equation of $U_\Gamma$ using the McKay quiver with relations $(Q, R)$.

Key words: McKay correspondence; $G$-Hilbert scheme; Quiver representations.

1 Introduction.

The generalisation of the McKay correspondence [8], [12] to small finite subgroups $G \subset \text{GL}(2, \mathbb{C})$ was established after Wunram [15] introduced the notion of special representation. The so-called “special” McKay correspondence relates the $G$-equivariant geometry of $\mathbb{C}^2$ and the minimal resolution $Y$ of the quotient $\mathbb{C}^2/G$, establishing a one-to-one correspondence between the irreducible components of the exceptional divisor $E \subset Y$ and the special irreducible representations. This minimal resolution $Y$ can be viewed as two equivalent moduli spaces: by a result of Ishii [4] it is known that $Y = G\text{-Hilb}(\mathbb{C}^2)$ the $G$-invariant Hilbert scheme introduced by Ito and Nakamura [5], and at the same time as $Y = \mathcal{M}_\theta(Q, R)$ the moduli space of $\theta$-stable representations of the McKay quiver.

In the same spirit as [7] in this paper we treat the problem of describing $G\text{-Hilb}(\mathbb{C}^2)$ by giving an explicit affine open cover. In [9] Nakamura introduced the notion of $G$-graphs, providing a nice and friendly framework to describe $G\text{-Hilb}(\mathbb{C}^2)$ for finite abelian subgroups in $\text{GL}(n, \mathbb{C})$. In this paper we consider the non-abelian analogue of a $G$-graph and provide an explicit method to interpret $\theta$-stable representations of the McKay quiver from $G$-graphs and vice versa. By using the relations on the McKay quiver, this led us to describe explicitly an open cover $\mathcal{M}_\theta(Q, R)$ (hence for $G\text{-Hilb}(\mathbb{C}^2)$) for binary dihedral subgroups in $\text{GL}(2, \mathbb{C})$ with the minimal number of open sets. Our method also recovers the ideals defining the $G$-clusters in $G\text{-Hilb}(\mathbb{C}^2)$. An alternative description of an open cover for the minimal resolution $Y$ has been discovered independently by Wemyss [13], [14] by using reconstruction algebras instead of the skew group ring.

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2 Preliminaries.

2.1 Dihedral groups $\text{BD}_{2n}(a)$ in $\text{GL}(2, \mathbb{C})$.

Let $G$ be a finite small binary dihedral subgroup in $\text{GL}(2, \mathbb{C})$. In terms of its action on the complex plane $\mathbb{C}^2$ we consider the representation of $G$, denoted by $\text{BD}_{2n}(a)$, generated by $\alpha = \left( \begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix} \right)$ and $\beta = \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ subject to the relations:

$$(2n, a) = 1, \quad a^2 \equiv 1 \pmod{2n}, \quad \gcd(a + 1, 2n) \nmid n$$

where $\varepsilon$ is a primitive $2n$-th root of unity. The group $\text{BD}_{2n}(a)$ has order $4n$ and it contains the maximal normal index 2 cyclic subgroup $A := \langle \alpha \rangle \triangleleft G$, which we denote by $\frac{1}{2n}(1, a)$ (note that $\beta^2 \in A$). The condition $a^2 \equiv 1 \pmod{2n}$ is equivalent to the relation $\alpha \beta = \beta \alpha^a$, and $\gcd(a + 1, 2n) \nmid n$ implies that the group is small (see [10], §3 for details).
Definition 2.1. Let \( q := \frac{2n}{(a-1,2n)} \), and \( k \) such that \( n = kq \).

The group \( BD_{2n}(a) \) has \( 4k \) irreducible 1-dimensional representations \( \rho^\pm_j \) and \( \rho^-_j \) of the form

\[
\rho^\pm_j(\alpha) = \varepsilon^j, \quad \rho^-_j(\beta) = \left\{ \begin{array}{ll} \pm i & \text{if } n,j \text{ odd} \\ \pm 1 & \text{otherwise} \end{array} \right.
\]

where \( \varepsilon \) is a \( 2n \)-th primitive root of unity and \( j \) is such that \( j \equiv aj \) (mod \( 2n \)). The values \( r \) for which \( r \neq ar \) (mod \( 2n \)) form in pairs \( n - k \) irreducible 2-dimensional representations \( V_r \) of the form

\[
V_r(\alpha) = \left( \begin{array}{cc} \varepsilon^r & 0 \\ 0 & \varepsilon^{ar} \end{array} \right), \quad V_r(\beta) = \left( \begin{array}{cc} 0 & 1 \\ (-1)^r & 0 \end{array} \right)
\]

By definition, the natural representation is \( V_1 \).

In what follows we take the notation as in [16] §10. Let \( V = V_1 \) a vector space with basis \( \{ x, y \} \) where \( G \) acts naturally. Define \( S = \text{Sym} V := \mathbb{C}[V^*] \) the polynomial ring in the variables \( x \) and \( y \). Then the action of \( G \) extends to \( S \) by \( g \cdot f(x,y) := f(g(x), g(y)) \) for \( f \in S, g \in G \).

Definition 2.2. Let \( G = BD_{2n}(a), f \in S \).

\[
f \in \rho^-_j \iff \alpha(f) = \varepsilon^j f, \beta(f) = \left\{ \begin{array}{ll} \pm if & \text{if } n,j \text{ odd} \\ \pm f & \text{otherwise} \end{array} \right.
\]

\[
(f, \beta(f)) \in V_k \iff \alpha(f, \beta(f)) = \left( \varepsilon^k f, \varepsilon^{ak} \beta(f) \right)
\]

Let \( S_\rho := \{ f \in \mathbb{C}[x,y] : f \in \rho \} \) the \( S^G \)-module of \( \rho \)-invariants. Note that these are precisely the Cohen Macaulay \( S^G \)-modules \( S_\rho = (S \otimes \rho^*)^G \) where \( G \) acts on \( S \) as above and \( G \) acts on a representation \( \rho \) by the inverse transpose.

### 2.2 G-Hilb and G-graphs.

Let \( G \subset \text{GL}(n, \mathbb{C}) \) be a finite subgroup.

Definition 2.3. A \( G \)-cluster is a \( G \)-invariant zero dimensional subscheme \( Z \subset \mathbb{C}^n \), defined by an ideal \( I_Z \subset \mathbb{C}[x_1, \ldots, x_n] \), such that \( O_Z = \mathbb{C}[x_1, \ldots, x_n]/I_Z \cong \mathbb{C}[G] \) the regular representation as \( G \)-modules. The \( G \)-Hilbert scheme \( \text{G-Hilb}(\mathbb{C}^n) \) is the moduli space parametrising \( G \)-clusters.

Recall that \( \mathbb{C}[G] = \bigoplus_{\rho \in Irr(G)} (\rho_i)^{\dim \rho_i} \), where every irreducible representation \( \rho_i \) appears (\( \dim \rho_i \)) times in the sum. Thus, as a vector space, \( O_Z \) has its basis (\( \dim \rho_i \)) elements in each \( \rho_i \). To describe a distinguished basis of \( O_Z \) with this property, it is convenient to use the notion of \( G \)-graph.

Definition 2.4. Let \( G \subset \text{GL}(n, \mathbb{C}) \) be a finite subgroup. A \( G \)-graph is a subset \( \Gamma \subset \mathbb{C}[x_1, \ldots, x_n] \) satisfying the following conditions:

1. It contains (\( \dim \rho \)) number of elements in each irreducible representation \( \rho \).

2. If a monomial \( x_1^{\nu_1} \cdots x_n^{\nu_n} \) is a summand of a polynomial \( P \in \Gamma \), then for every \( 0 \leq \mu_j \leq \lambda_j \) the monomial \( x_1^{\mu_1} \cdots x_n^{\mu_n} \) must be a summand of some polynomial \( Q_{\mu_1, \ldots, \mu_n} \in \Gamma \).

Note that for any \( G \)-cluster \( Z \) we can choose a basis for the vector space \( O_Z \) which is a \( G \)-graph. For example, let \( f, g \in \rho \) with \( \rho \) a 1-dimensional representation and suppose \( f \notin \Gamma \) and \( g, xg \in \Gamma \). Then \( f = \alpha g \) and \( xg = \alpha(xg) \) (mod \( I \)) for some \( \alpha \in \mathbb{C} \). Since \( xg \in \Gamma \) we have \( \alpha \neq 0 \) which imply that we can choose \( f \in \Gamma \) (see [7], Chapter 2 §2.3).

For any \( G \)-graph \( \Gamma \) there exists an open set \( U_\Gamma \subset \text{G-Hilb}(\mathbb{C}^n) \) consisting of all \( G \)-clusters \( Z \) such that \( O_Z \) admits \( \Gamma \) as a basis for \( O_Z \). Therefore, given the set of all possible \( G \)-graphs \( \{ \Gamma_i \} \), their union covers \( \text{G-Hilb}(\mathbb{C}^n) \).

Example 2.5. \( \Gamma = \{ 1, x, x^2, y, xy \} \) is a \( 1/4(1,3) \)-graph. For the non-abelian binary dihedral group \( D_4 = \langle \frac{1}{2}(1,3), 0, 1 \rangle \subset \text{SL}(2, \mathbb{C}) \), \( A = \{ 1, x, y, x^2 + y^2, x^2 - y^2, y^3, -x^3, x^4 - y^4 \} \) is a \( D_4 \)-graph (note that \( (x, y), (y^3, -x^3) \in V_1 \)).
The representation of a $G$-graph $\Gamma$ consist of the set of monomials which are summands of polynomials in $\Gamma$. Let $I_\Gamma$ be the ideal generated by every polynomial $f \in \rho$, for some irreducible representation $\rho$, which is not in $\Gamma$. We say that the $G$-graph $\Gamma$ is represented by the ideal $I_\Gamma$.

In Example 1, the representations of $\Gamma$ and $\Lambda$ are shown in Fig. 1. The $G$-graph $\Gamma$ is represented by the ideal $I_\Gamma = (x^3, x^2y, y^3)$. Similarly, $\Lambda$ is represented by the ideal $I_\Lambda = (xy, x^4 + y^2)$ where the elements $x^3 + y^3 \in \rho^+\zeta_2$ and $x^2 - y^2 \in \rho^+\zeta_2$ are represented by $x^2$ and $y^2$ respectively. The relation $x^3 + y^3 = 0$ identifies $x^3$ and $y^3$ in $\mathbb{C}[x,y]/I_\Lambda$.

![Figure 1: Representation of the $G$-graphs $\Gamma$ and $\Lambda$.](image)

3 \textbf{ $G$-graphs for $BD_{2n}(a)$ groups.}

Let $G = BD_{2n}(a) \subset GL(2, \mathbb{C})$. The minimal resolution $Y$ of $\mathbb{C}^2/G$ is obtained as follows (see [5] §1.2): First act with $A$ on $\mathbb{C}^2$ and consider $A$-$\text{Hilb}(\mathbb{C}^2)$ as the minimal resolution of $\mathbb{C}^2/A$. To complete the action of $G$ act with $G/A$ (generated by the class of $\beta$) on $A$-$\text{Hilb}(\mathbb{C}^2)$. The conditions $a^2 \equiv 1 \pmod{2n}$ and $\gcd(a+1, 2n) \mid n$ imply that the continued fraction $\frac{a}{n} \beta$ is symmetric with respect to the middle entry. Therefore the coordinates along the exceptional divisor $E = \bigcup_{i=1}^{2n-1} E_i \subset A$-$\text{Hilb}(\mathbb{C}^2)$ are symmetric with respect to the middle curve $E_m$. The action of $G/A$ identifies the rational curves on $E$ pairwise except in $E_m$ where we have an involution. Thus the quotient $\tilde{Y} = A$-$\text{Hilb}(\mathbb{C}^2)/(G/A)$ has two $A_1$ singularities, and the blow-up of these two points gives us $G$-$\text{Hilb}(\mathbb{C}^2)$ by the uniqueness of minimal models of surfaces.

Let us now translate this construction into graphs. Any orbit of $G/A$ in $A$-$\text{Hilb}(\mathbb{C}^2)$ consists of two $A$-clusters $Z$ and $\beta(Z)$, with symmetric $A$-graphs $\Gamma$ and $\beta(\Gamma)$ respectively. Then $\Gamma$ is represented by the monomial ideal $I_\Gamma = (x^r, y^u, x^{u-r}y^{u-r})$ and $I_{\beta(\Gamma)} = (y^r, x^u, x^{u-r}y^{u-r})$, where $e_i = \frac{1}{2n}(r,s)$, $e_{i+1} = \frac{1}{2n}(u,v)$ are two consecutive lattice points in the boundary of the Newton polygon of the lattice $L := \mathbb{Z}^2 + \frac{1}{2n}(1,a) \cdot \mathbb{Z}$.

If we denote by $\mathcal{Y}$ the corresponding $G$-cluster, then it is clear that $\mathbb{C}[x,y] \supset \mathcal{Y}$. Thus $I_{\mathcal{Y}} \subset I_{\mathbb{C}[x,y]}$. Note that the representation of $\tilde{\Gamma}$ in the lattice of monomials is symmetric with respect to the diagonal, and the inclusion $\tilde{\Gamma} \subset \mathcal{Y}$ is never an equality since $\Gamma$ and $\beta(\Gamma)$ always share a common subset of elements $R \subset \Gamma$. The subset $\tilde{\Gamma}$ is called $qG$-graph.

Thus, to obtain a $G$-graph from $\tilde{\Gamma}$ we must add $xR$ elements to $\tilde{\Gamma}$ preserving the representation spaces contained in $R$ according to Definition 2.4. It is shown in [11] that this extension process from a $qG$-graph $\tilde{\Gamma}$ to a $G$-graph $\Gamma$ is unique. It leads to the following theorem, which resumes the classification of $G$-graphs for $BD_{2n}(a)$ groups describing their defining ideals in each case.

\textbf{Theorem 3.1 ([11])}. Let $G = BD_{2n}(a)$ be a small binary dihedral group and let $\Gamma_1$ be the $A$-graph corresponding to the two consecutive lattice points $e_i = \frac{1}{2n}(r,s)$, $e_{i+1} = \frac{1}{2n}(u,v)$ of the Newton polygon of the lattice $L$. Denote by $\Gamma := \Gamma_1 \cup \beta(\Gamma_1)$ and $\Gamma_2 := \Gamma \cup \beta(\Gamma)$. Then we have the following possibilities:

- If $u < s - v$ then $\Gamma$ is of type $A$ and it is represented by the ideal $I_A = (x^u y^u, x^r y^{u-r} + (-1)^{u-r} x^{u-r} y^{u-r}, x^{r+s} + (-1)^r y^{r+s})$.

- If $u - r = s - v := m$ then $\Gamma$ is of type $B$ and
  \begin{itemize}
  \item (a) If $u < 2m$ then $\Gamma$ is of type $B_1$ and it is represented by the ideal $I_{B_1} = (x^{r+s} + (-1)^r y^{r+s}, x^{m+s} y^{m-r} + (-1)^{m-r} x^{m-r} y^{m+s}, x^u y^m, x^m y^u)$. \\
  \item (b) If $u \geq 2m$ then $\Gamma$ is of type $B_2$ and $I_{B_2} = (x^{2m} y^m, x^{s+m}, y^{s+m}, x^u y^m, x^m y^u)$.
  \end{itemize}
In addition, when \( u = v = q := \frac{2n}{(a - 1, 2n)} \) we have four \( G \)-graphs of types \( C^+, C^-, D^+ \) and \( D^- \).

- The \( G \)-graphs of types \( D^\pm \) are represented by the ideals \( I_{D^\pm} = (x^q \pm (i)^q y^q, x^{r-q} y^{s-r}) \).
- For \( G \)-graphs of types \( C^\pm \) we have two cases:
  
  (a) If \( 2q < s \), and we call \( m_1 := s - q \) and \( m_2 := q - r \), they are represented by the ideals
  
  \[
  I_{C^+_B} = ((x^s \pm (i)^q y^q)^2, x^s y^m \pm (-1)^r i^q x^m y^s, x^m y^m \pm (-1)^m x^m y^m).
  \]
  
  (b) If \( 2q = r + s \) then \( I_{C^-_B} = (y^m (x^2 \pm (i)^q y^q), x^m (x^2 \pm (i)^q y^q), x^s y^r, x^m y^m) \).

**Remark 3.2.** The list of ideals in Theorem 3.1 define in \( G\text{-Hilb}(\mathbb{C}^2) \) the intersection points of two of the exceptional curves plus the strict transform of the coordinate axis in \( \mathbb{C}^2 \).

**Example 3.3.** Consider the two symmetric \( \Gamma \)-graphs given by \( I_\Gamma = (x^7, y^2, x^3 y) \) and \( I_{\Gamma_\Gamma} = (y^7, x^2, x^5 y) \), with \( r = 1, s = 7, u = 2, v = 2 \). The overlap subset is \( R = \{1, x, y, xy\} \) where \( 1 \in \rho_0^+, x, y \in \rho_\pm \), and \((x, y) \in V_1\). Then we must add the elements \( x^2 y - y^2 x \in \rho_0^+, x^8 - y^8 \in \rho_8 \) and the pair \((y^7, -x^7) \in V_1\). The resulting \( G \)-graph is represented by the ideal \((x^2 y^3, x^3 y^4 - y^8, -x^8 - y^8)\).

**Theorem 3.4 ([11]).** Let \( G = BD_{2n}(a) \) be small and let \( P \in G\text{-Hilb}(\mathbb{C}^2) \) be defined by the ideal \( I \). Then we can always choose a basis for \( \mathbb{C}[x, y]/I \) from the list \( \Gamma_A, \Gamma_B, \Gamma_C, \Gamma_C, \Gamma_D, \Gamma_D \). Moreover, if \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}, \Gamma_C, \Gamma_C, \Gamma_D, \Gamma_D \) is the list of \( G \)-graphs, then

\[
U_{\Gamma_0}, U_{\Gamma_1}, \ldots, U_{\Gamma_{m-1}}, U_{\Gamma_C}, U_{\Gamma_{m-1}}, U_{\Gamma_D}, U_{\Gamma_D}
\]

form an open cover of \( G\text{-Hilb}(\mathbb{C}^2) \).

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**4. \( M_\theta(Q, R) \) and Orbifold McKay quiver.**

Let \( G = BD_{2n}(a) \) and let \( A = \frac{1}{2}(1, a) \triangleleft G \). Denote by \( \text{Irr} G \) the set of irreducible representations of \( G \) (similarly for \( \text{Irr} A \)). For the background material on quivers refer to [1]. We consider left modules (and actions), and by a path \( pq \) we mean \( p \) followed by \( q \). Let \( (Q, R) \) a quiver with relations, fix \( \mathbf{d} = (d_i)_{i \in \mathbb{Q}_a} \) the dimension vector of the representations of \( (Q, R) \), and let \( \forall(I_R) \subset \mathbb{A}_N^\theta \cong \bigoplus_{\alpha \in Q_0} \text{Mat}_{d_i d_i}(\mathbb{C}) \) the representation space subject to the ideal of relations \( I_R \). For \( \theta \) generic we define \( M_\theta(Q, R) := \forall(I_R)/\theta \prod GL(d_i) \), the moduli space of \( \theta \)-stable representations of \( (Q, R) \) (see [6], [3] for details). Taking \( Q \) to be the McKay quiver and a particular choice of generic \( \theta \) (see §5) it is well known that \( M_\theta(Q, R) \cong G\text{-Hilb}(\mathbb{C}^2) \).

The McKay quiver of \( G \), denoted by \( \text{McKay}(Q, \theta) \), is defined by having one vertex for every \( \rho \in \text{Irr} G \) and by the number of arrows from \( \rho \) to \( \sigma \) to be \( \text{dim}_G \text{Hom}_G(\rho, \sigma \otimes V) \). Equivalently, due to Auslander it is known that McKay\( Q, \theta \) is the underlying quiver of the algebra \( \text{End}_{\mathbb{C}[x, y]}(\bigoplus_{\rho \in \text{Irr} \theta} S_\rho) \) where \( S_\rho = (S \otimes \rho)^G \) as in 2.2 (see [16] for a proof in dimension 2).

The McKay\( Q, \theta \) can be drawn as a quiver as follows: Let \( M \cong \mathbb{Z}^2 \) be the lattice of monomials and \( M_{\text{inv}} \cong \mathbb{Z}^2 \) the sublattice of invariant monomials by \( A \). If we take \( M_\mathbb{Z} = M \otimes \mathbb{Z} \) we can consider the torus \( T := M_\mathbb{Z}/M_{\text{inv}} \). The vertices in \( \text{McKay}(A) \) are precisely \( Q_0 = M \cap T \), and the arrows between vertices are the natural multiplications by \( x \) and \( y \) in \( M \). It is easy to see that we can always choose a fundamental domain \( D \) for \( T \) to be the parallelogram with vertices \( (0, 0), (2q, 0), (k+2q, k) \) where the opposite sides are identified.

**Proposition 4.1.** (i) The McKay quiver \( Q \) of the binary dihedral group \( BD_{2n}(a) \) is the \( \mathbb{Z}/2 \)-orbifold quotient of the McKay quiver for the Abelian subgroup \( A = \frac{1}{2}(1, a) \) (see Figure 2).

(ii) The relations \( R \) on the \( Q \) which gives the identification between \( G\text{-Hilb}(\mathbb{C}^2) \) and \( M_\theta(Q, R) \) are:

\[
\begin{align*}
  a_i b_i = 0, & f_i e_i = 0 \\
  c_i d_i = 0, & h_i g_i = 0 \\
  b_i a_i + d_i c_i = r_{i,1} u_{i,1} \\
  e_i f_i + g_i h_i = u_{i,q-2} r_{i+1,q-2} \\
  u_{i,j} r_{i+1,j} = r_{i,j+1} u_{i,j+1}
\end{align*}
\]

where we consider the subindices modulo \( k \).
Notation 4.2. The source and target for $r_{i,j}$ and $u_{i,j}$ are $r_{i,j} : S_{V_{[i-(i+1)+j]}-i} \to S_{V_{[i-(i+1)+j+1]}+i}$, and $u_{i,j} : S_{V_{[i-(i+1)+j]}+j-i} \to S_{V_{[i-(i+1)+j]+1}}$ with $i \in \{0, k-1\}$, $j \in \{1, q-2\}$, where $\tilde{i}$ denotes $i \mod k$.

**Remark 4.3.** In the case $q = 2$ the relations are $a_i b_i+1 = 0$, $f_i e_i+1 = 0$, $c_i d_i+1 = 0$, $h_i g_i+1 = 0$ and $b_i a_i + d_i c_i = e_i f_i + g_i h_i$.

**Remark 4.4.** Our choice of modules $S_{\rho} := (S \otimes \rho^*)^G$ imply that the quiver in Figure 2 has opposite arrows as in [2]. Note that $V_1 \to \rho^+ \to V_1^*$ while $S_{V_1^*} \to S_{\rho^+} \to S_{V_1}$. Since it is just a matter of convention, the relations do not change.

![Figure 2: McKay quiver for BD_{2n}(a) groups](image)

**Proof.** (i) Let $\text{Irr } A = \{\rho_0, \ldots, \rho_{2n-1}\}$. The group $G$ acts on $A$ by conjugation, which induces an action of $G/A \cong \mathbb{Z}/2$ on $A$ by $\beta \cdot h := \beta h^{-1} \beta$, for any $h \in A$. Therefore $G/A$ acts on $\text{Irr } A$ by $\beta \cdot \rho_k := \rho_{k \beta}$, for $\rho_k \in \text{Irr } A$. The free orbits under this action are $\{\rho_i, \rho_{\alpha i}\}$ with $\alpha \neq i \mod 2n$, and they produce the two 1-dimensional representations $\rho_i^+$ and $\rho_i^-$ in $\text{MckayQ}(G)$.

The fixed representations are contained in the left (and identified right) side of $\mathcal{D}$, and in the line parallel to it passing through the middle of $\mathcal{D}$ (more precisely, $\rho_{i(1+a)}$ and $\rho_{j(1+a)+q}$ for $0 \leq i < k$ respectively). Note that $\text{MckayQ}(G)$ is now drawn on a cylinder where only the top and bottom sides are identified. The arrows of $\text{MckayQ}(A)$ going in and out of representations in the fixed locus, split into two different arrows, while for the rest we have a one-to-one correspondence between arrows in $\text{MckayQ}(A)$ and $\text{MckayQ}(G)$.

(ii) Let $kQ$ be the path algebra, $S = \mathbb{C}[x_1, \ldots, x_r]$ and $V = V_1$ the natural representation. Tensoring with $\text{det}_V := \wedge^\tau V$ induces a permutation $\tau$ on the vertices of $Q$ as follows: $e_i = \tau(e_j) \iff \rho_i = \rho_j \otimes \text{det}_V$. Now consider an arrow $a : e_i \to e_j$ as an element $\psi_{\alpha} \in \text{Hom}_{\mathbb{C}G}(\rho_{i}, \rho_{j} \otimes V)$. Then for any path $p = a_1 a_2 \cdots a_r$ of length $r$ we can consider the $G$-module homomorphism

$$
\rho_{t(p)} \xrightarrow{\psi_p} \rho_{h(p)} \otimes V^\otimes \tau \xrightarrow{\text{id}_{h(p)} \otimes \gamma} \rho_{h(p)} \otimes \text{det}_V
$$

(1)

The $\psi_p$ is the composition of the maps $\psi_{a_1}$ and $\psi_{a_i} \otimes \text{det}_V$ for $i = 2, \ldots, r$, and $\gamma : V^{\otimes r} \to \bigwedge^{\tau} V$ sends $v_1 \otimes \cdots \otimes v_r \mapsto v_1 \wedge \cdots \wedge v_r$. By Schur’s Lemma the composition of maps in (1) is zero if $\tau(h(p)) \neq t(p)$, a scalar $c_p$ otherwise. It is known by [2] that for a finite small $G \subset \text{GL}(r, \mathbb{C})$ the skew group algebra $S^G$ is Morita equivalent to the algebra $kQ/\{\partial_q \Phi : |q| = r - 2\}$, where $\Phi := \sum_{|q|=r-2} (c_q \dim h(p)) p$ and $\partial_q$ are derivations with respect to paths of length $|q|$. Since the $\theta$-stable $S^G$-modules ($\theta$ as in §5) are precisely the $G$-clusters, this relations gives $M_{t}(Q, R) \cong G\text{-Hilb} (C^\tau)$.

For the groups $G = BD_{2n}(a)$ in dimension 2, tensoring with $\text{det}_V = \rho_{1+1}^+$ induces a permutation $\tau$ on $\text{MckayQ}(G)$ which translates it one step diagonally down, e.g., $\text{det}(\rho^+) = V^*$ (or the equivalently diagonally up in Figure 2, e.g. $\text{det}(S_{\rho^+}) = S_{V^*}$). Only paths of length 2 joining two vertices identified by $\tau$ appear in $\Phi$, giving the relations $R$ by derivations with respect to the vertices of $Q$.  

\[ \square \]
Example 4.5. Consider the group BD$_{30}(19)$ generated by $\alpha = \text{diag}(\varepsilon, \varepsilon^{19})$ with $\varepsilon$ a primitive 30-th root of unity, and $\beta = (\begin{smallmatrix} 0 & 1 \\ \frac{1}{\varepsilon} & 0 \end{smallmatrix})$. We have $q = 5$ and $k = 3$. The continued fraction $\frac{30}{19 - 19} = \frac{30}{11} = [3, 4, 3]$ describes the lattice $M_{\text{inv}}$. The two consecutive invariant monomials $x^3y^3$ and $x^4y$ define a fundamental domain of the lattice $T$, which can be translated into the parallelogram filled with numbers shown in Figure 3 (a). The diagram represents the lattice $M$ where the bottom left corner represents the monomial 1 and the numbers denote the representation to which they belong to, e.g. the monomial 0 corresponds to monomials in $M_{\text{inv}}$. Opposite sides of the parallelogram are identified. The McKay quiver is completed by adding at every vertex the two arrows corresponding to the multiplication by $x$ and $y$ to the corresponding adjacent vertices.

Now acting by $\beta$ we see that representations $\rho_0, \rho_20, \rho_10$ and $\rho_5, \rho_{25}, \rho_{15}$ are fixed, while the rest (in pairs) are contained in a free orbit. The McKay $Q$ of $\text{BD}_{30}(19)$ is also shown in Figure 3 (b). Notice that top and bottom rows are identified.

$$
\begin{array}{c}
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 \\
\end{array}
\end{array}
$$

Figure 3: McKay quivers for (a) the abelian group $\frac{1}{30}(1, 19)$ and for (b) the group $\text{BD}_{30}(19)$.

5 Explicit calculation of $G$-Hilb($\mathbb{C}^2$)

Let $G = \text{BD}_{2n}(a)$ and $(Q, R)$ be the McKay quiver as in 4. Denote the arrows by $a = (a, A)$, or by $a = (a')^\dagger$ depending on the dimensions of the vector spaces at the source and target of $a$. We consider representations of $Q$ with dimension vector $d = (\dim \rho_i)_{i \in Q_0}$ and the generic stability condition $\theta = (1 - \sum_{i \in Q_0} \dim \rho_i, 1, \ldots, 1)$, which imply $G$-Hilb($\mathbb{C}^2$) $= M_{\theta}(Q, R)$.

Claim 5.1. A representation of $Q$ is stable if and only if there exist $\dim \rho_i$ linearly independent paths from the distinguished source, chosen to be $\rho_0 \in Q_0$, to every other vertex $\rho_i$ in $Q$.

Indeed, a representation $W$ is not $\theta$-stable if $\exists W' \subset W$ proper with $\theta(W') < 0$. Since the only nonzero entry of $\theta$ corresponds to $\rho_0^\dagger$ and the dimension vector $d(W')$ has to be strictly smaller than $d(W)$, this is equivalent to say that there are strictly less linearly independent paths from $\rho_0^\dagger$ to $\rho_i$ than $\dim \rho_i$, for $i \neq 0$.

By making a correspondence between elements of a $G$-graph and paths in $Q$, an open cover of $M_{\theta}(Q, R)$ is given by the ones corresponding to the $G$-graphs. The $G$-graphs predetermine the choices of linear independent paths, thus giving a covering of $M_{\theta}(Q, R)$ with the minimal number of open sets.

Theorem 5.2. Let $G = \text{BD}_{2n}(a)$ and $\Gamma = \Gamma(r, s, u, v)$ be a $G$-graph with corresponding open set $U_{\Gamma} \subset M_{\theta}(Q, R)$. Then,

- If $\Gamma$ is of type $A$ then $U_{\Gamma, A}$ is given by:
  
  $a_0, D_0, F_0 \neq 0$, and $e_i, g_i, r_{i,j}, U'_{i,j} \neq 0$ for all $i, j$,
  
  $a_i, H_i \neq 0$ for $i$ even, and $e_i, F_i \neq 0$ for $i$ odd.
  
  For $0 < i < u$ set $b_i, D_i \neq 0$ if $i$ is even, and $B_i, d_i \neq 0$ if $i$ is odd.
  
  For $i \geq u$ set $B_i, D_i \neq 0$.

- If $\Gamma$ is of type $B$ then $U_{\Gamma, B}$ is given by:
  
  $a_0, d_0, H_0 \neq 0$, $e_i \neq 0$, $g_i \neq 0$, $r_{i,j} \neq 0$ for all $i, j$,
  
  $a_i, b_i, D_i, H_i \neq 0$ for $i$ even, and $B_i, c_i, d_i, F_i \neq 0$ for $i$ odd.

If $r > 1$ also $C_0, R_{1,1}', \ldots, R_{1,r-2}' \neq 0$. 

6
If \( \Gamma \) is of type \( B_1 \) then also set
\[
R_{r+1}^{1,1}, \ldots, R_{r+1,r-r-2}^{1,1}, U_{r+1}^{1,1} \neq 0 \text{ and } U_{r+1}^{1,1} \neq 0, \forall i \neq 0, r \text{ and } \forall j.
\]
For \( i = 0, r \) set \( U_0^{r,r}, \ldots, U_{0,q-2}^{r,r} \neq 0 \) and \( U_{r,r-2}^{r,r}, \ldots, U_{r,q-2}^{r,r} \neq 0. \)
Also if \( q > 2, C_r \neq 0 \) if \( r \) even, or \( A_r \neq 0 \) if \( r \) odd.

If \( \Gamma \) is of type \( B_2 \) then also set \( U_{r+1}^{1,1} \neq 0, \forall i \neq 0 \) and \( U_{r+1,0}^{0,0}, \ldots, U_{r+1,q-2}^{0,0} \neq 0. \)

1. If \( \Gamma \) is of type \( C \) then
   (a) The conditions for \( U_{1}^{r} \subset \mathcal{M}_Q(Q, R) \) are the same as those for \( \Gamma_i \) for \( i = A \) or \( B \), and the condition \( F_0 = 0 \) instead of \( H_0 = 0. \)
   (b) The open conditions for the case \( \Gamma^C \) are the same as those for \( \Gamma^C \) but swapping the conditions for \( F_i \) for \( H_i \) and vice versa.
2. If \( \Gamma \) is of type \( D \) then \( U_{1}^{r} \subset \mathcal{M}_Q(Q, R) \) is defined by:
\[
\begin{align*}
& a_0, C_0, d_0 \neq 0, \text{ and } a_i, b_i, D_i \neq 0 \text{ for } i \text{ even}, B_i, c_i, d_i \neq 0 \text{ for } i \text{ odd}, \\
& U_{i,j}^{r} \neq 0 \text{ for all } i > 0 \text{ and all } j, U_{0,r}^{r}, \ldots, U_{0,q-2}^{r} \neq 0, \\
& r_{i,j} \neq 0 \text{ for all } i, j \text{ except for } r_{i,q-i}, i \in [2, k-1], \\
& R_{r+1}^{1,1}, \ldots, R_{r+1,r-r-2}^{r+1,1} \neq 0, u_{i,q-1} \neq 0 \text{ for } i \in [2, k-1].
\end{align*}
\]

If \( \Gamma \) is a \( G \)-graph of type \( D^+ \) then we also set
\[
E_0, H_0, G_0, E_1, f_1, g_1, H_1 \neq 0.
\]
If \( i \) is even then \( E_i, g_i, F_i \neq 0 \), if \( i \) is odd then \( e_i, G_i, H_i \neq 0. \)

If \( \Gamma \) is a \( G \)-graph of type \( D^- \) we set
\[
E_0, F_0, g_1, c_1, F_1, G_1, h_1 \neq 0.
\]
If \( i \) is even then \( e_i, G_i, H_i \neq 0 \), if \( i \) is odd then \( e_i, G_i, F_i \neq 0 \text{ with } i \in [0, k-1]. \)

Proof. An open set in \( \mathcal{M}_Q(Q, R) \) is obtained by making open conditions in the parameter space of representations \( \mathcal{V}(I_R) \subset \mathbb{A}^N. \) We can change basis at every vertex to chose 1 as basis for every 1-dimensional vertex, and (1,0) and (0,1) for every 2-dimensional. Thus, by 5.1 the element 1 \( \in \mathbb{A}^+ \) generates the whole representation with this basis. For instance, we always choose \( a_0 = (1,0). \)

Given any \( G \)-graph \( \Gamma \), the corresponding open set \( U_{1}^{r} \subset \mathcal{M}_Q(Q, R) \) is obtained by taking the open conditions according to the elements of \( \Gamma \). This is done by considering simultaneously the McKay quiver of \( G \) to be given (see 4) by the \( S \)-modules \( S_p \) as vertices, and the irreducible maps between them to be the arrows. See Figure 4 for the case \( n \) even, where the segment is repeated throughout the quiver. When \( n \) is odd replace \( e_i \) by \( r_i \) and \( f_i \) by \( y_i \). We can change basis at every vertex to chose 1 as basis for every 1-dimensional vertex, and (1,0) and (0,1) for every 2-dimensional. Thus, by 5.1 the element 1 \( \in \mathbb{A}^+ \) generates the whole representation with this basis. For instance, we always choose \( a_0 = (1,0). \)

Given any \( G \)-graph \( \Gamma \), the corresponding open set \( U_{1}^{r} \subset \mathcal{M}_Q(Q, R) \) is obtained by taking the open conditions according to the elements of \( \Gamma \). This is done by considering simultaneously the McKay quiver of \( G \) to be given (see 4) by the \( S \)-modules \( S_p \) as vertices, and the irreducible maps between them to be the arrows. See Figure 4 for the case \( n \) even, where the segment is repeated throughout the quiver. When \( n \) is odd replace \( e_i \) by \( r_i \) and \( f_i \) by \( y_i \). We can change basis at every vertex to chose 1 as basis for every 1-dimensional vertex, and (1,0) and (0,1) for every 2-dimensional. Thus, by 5.1 the element 1 \( \in \mathbb{A}^+ \) generates the whole representation with this basis. For instance, we always choose \( a_0 = (1,0). \)

Irreducible maps send 1 \( \in S_{p} \) once to every other \( S_{p} \) and twice to every other \( S_{p} \) linearly independently. Denote the polynomials obtained by \( f_{p} \) and \( (g_{r}, g_{r}'), (h_{r}, h_{r}') \) respectively. In this way, for any stable representation all modules \( S_p \) have assigned basis polynomials. Therefore, if we take the open conditions such that the basis elements generated from 1 \( \in S_{p} \) form the \( G \)-graph \( \Gamma \), we obtain the desired open set \( U_{1}^{r} \subset \mathcal{M}_Q(Q, R). \)

If \( f \in S_{p} \) and \( f \neq f_{p} \) (i.e. \( f \notin \Gamma \)), then there exists \( c \in \mathbb{C} \) such that \( f = cf_{p} \).
(f, f') ∈ S_v). Therefore $U_T$ parametrizes every G-cluster with $\Gamma$ as G-graph, so the union of $U_T$ covers $\mathcal{M}_G(Q, R)$. We prove the result case by case. It is worth mention that because $G = BD_{2n}(a)$ we have that $\langle k, q \rangle = 1$ (see [10] §3.3.1).

**Case A:** We start to generate the representation from $r = 0$ and $a_0 = (1, 0)$. We choose to obtain the basis element $(1, 0)$ at every 2-dimensional vertex with horizontal arrows taking $r_i, j = \left( \begin{array}{c} 1 \\ r_i, j \end{array} \right)$ for all $i, j$. The open conditions needed are $r_i, j \neq 0 \forall i, j, a_i \neq 0$ for all $i$ even, $c_i \neq 0$ and $c$ odd. Similarly, we choose to reach $(0, 1)$ at every 2-dimensional vertex with vertical arrows taking $u_i, j = \left( \begin{array}{c} 0 \\ u_i, j \end{array} \right)$ for all $i, j$. We can achieve such a map with a map that has $d_1 = b_2 = d_3 = \ldots$ until $d_{u-1} = 1$ if $u$ is even, or $b_{u-1} = 1$ if $u$ is odd. The condition $x^u y^u \not\in \mathcal{G}_A$ is given by $B_u = 1$ if $u$ is even, or $B_u = 1$ if $u$ is odd. Finally, from row $u$ to the top row the choices are always $B_i, D_i \neq 0, i \neq 0$ and $D_0 \neq 0$.

**Case B:** In this case $x^u y^u, x^y \not\in \mathcal{G}_B$, which implies that $x^y \in \mathcal{G}_B$ for $i < u$. This explains the choices at the left hand side of the quiver, while on the right hand side remain the same as before. Since $x^u y^u, x^y \in \mathcal{V}_r$, the conditions $x^u y^u, x^y \not\in \mathcal{G}_B$ are expressed with choices $C_0, R_{1,1}, R_{1,2}, \ldots, R_{u-2} \neq 0$. If $r \leq k$ we have a G-graph of type $B_1$, otherwise we have a type $B_2$.

**Case C:** If the G-graph $\Gamma(r, s, u, v)$ is of type $B$, then the open conditions are made at the special representation $\mathcal{V}_r$. The difference between the $C^+$ and $C^-$ is given by $\langle + \rangle^2 \not\in \mathcal{G}_C^+$ and $\langle - \rangle^2 \not\in \mathcal{G}_C^-$ which are the choices on the vertical arrows in the right side of the quiver.

**Case D:** In this case $\langle + \rangle \not\in \mathcal{G}_D^+$, or $\langle - \rangle \not\in \mathcal{G}_D^-$. The open condition is made at the special representation $\rho_q^+$ (or at $\rho_q^-$ respectively). For instance, in the $D^+$ case we do not allow a path of length $q$ starting from $\rho_q^+$ and ending at $\rho_q^-$, i.e., $E_1 = 1$.

**Example 5.3.** Let $G = BD_{12}(7)$ with $q = 2, k = 3$. The G-graphs is shown in Figure 5. The representation spaces of $U_{\Gamma_A}$ and $U_{\Gamma_{C^-}}$ are given in Figure 6. The open choices are shown red colour and the red arrows represent the choices of the basis $(1, 0), (0, 1)$ at each 2-dimensional vertex.

In all of the cases, after doing the open choices and using the relations we obtain the whole representation space in terms of three parameters subject to a single relation, thus obtaining a hypersurface in $\mathbb{C}^3$. See Figure 6 where in the case $U_{\Gamma_A}$ we have $C_0 = c_0 d_1$, $a_1 = G_1 d_1 = 1$, and in the case $U_{\Gamma_{C^-}}$ we...
have \( a_1 = G_1 - D_1, c_1 = -b_2D_1, H_0 = b_2 - 1 \). The equations of the open cover of \( BD_{12}(7)\)-Hilb(\(\mathbb{C}^2\)) are:

\[
\begin{align*}
U_A : & \quad c_0d_1 = (c_0d_1^2 + 1)G_1, \\
U_C^- : & \quad b_2D_1 = -(b_2 - 1)G_1, \\
U_{D^+} : & \quad c_1f_0 = -(c_1^2f_0 - 1)D_1, \\
U_{C^+} : & \quad b_2D_1 = -(b_2 - 1)E_1, \\
U_{D^-} : & \quad g_1h_0 = -(g_1h_0^2 - 1)D_1.
\end{align*}
\]

Using the quiver in Figure 4 we can calculate the basis polynomials in every irreducible representation (see case \(\Gamma_D^-\) in Table 1), and we can deduce the gluings between every open piece. For instance, between \(U_{\Gamma_A}\) and \(U_{\Gamma_{C^-}}\) we have \((b_2, D_1, G_1) \mapsto (1 - c_0d_1^2, d_1^{-1}, G_1)\), i.e. they cover a \(-2\)-curve. As a result we obtain the following expected dual graph of the exceptional divisor in \(G\)-Hilb(\(\mathbb{C}^2\)):

![Diagram](image)

Figure 6: Open sets \(U_{\Gamma_A}\) and \(U_{\Gamma_{C^-}} \subset M_\theta(Q, R)\) for \(BD_{12}(7)\).

<table>
<thead>
<tr>
<th>(\mathcal{E})</th>
<th>(\rho_0^0)</th>
<th>(\mathcal{F})</th>
<th>(\rho_0^+)</th>
<th>(\mathcal{V})</th>
<th>(\rho_1)</th>
</tr>
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<tr>
<td>(\rho_0^-)</td>
<td>(2xy(+)^2)</td>
<td>(\rho_4^-)</td>
<td>(2xy(+)^3)</td>
<td>(V_1)</td>
<td>((0, 1) = (xy^2, y^2))</td>
</tr>
<tr>
<td>(\rho_4^-)</td>
<td>(2xy(+)^3)</td>
<td>(\rho_4^-)</td>
<td>(2xy(+)^4)</td>
<td>(V_1)</td>
<td>((0, 1) = (xy^2, y^3))</td>
</tr>
<tr>
<td>(\rho_2^-)</td>
<td>(2xy(+)^2)</td>
<td>(\rho_8^-)</td>
<td>(2xy)</td>
<td>(V_5)</td>
<td>((1, 0) = (xy^4, y^2))</td>
</tr>
<tr>
<td>(\rho_8^-)</td>
<td>(2xy(+)^4)</td>
<td>(\rho_{10}^-)</td>
<td>(2xy(+)^5)</td>
<td>(V_5)</td>
<td>((1, 0) = (xy^4, y^3))</td>
</tr>
</tbody>
</table>

Table 1: Basis elements of the \(G\)-graph \(\Gamma_{D^-}(1, 7, 2, 2)\).

References


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