Exact-categorical properties of subcategories of abelian categories

(Part I : General theory)

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§0.

- Exact cat : introduced by Quillen in 1973 [Higher algebraic K-theory]
- Exact cat $\mathcal{E}$ = additive cat + extra str $\mathbb{E}$
  ($\mathbb{E}$: class of "short exact seq in $\mathcal{E}$")
- Extension-closed subcat of an abelian cat has a natural exact str.
  (but exact str are not uniquely determined by $\mathcal{E}$)

Q. Why Exact Category?

A0. To define higher algebraic K-grp.

A1. It provides a framework for doing homological alg for (not necessarily abelian) additive cat.
(e.g. Banach sp, ...)

A2. Particular exact cat (Frobenius) gives a triangulated cat.
(called "algebraic tri. cat")
(e.g. homotopy cat of abelian cat)

My answer.

(*)

To study <u>subcat of an abelian cat!</u>

In fact, by regarding (*) as exact cat, we can consider properties and invariants of (*)

This gives us a lot of strategy & problem when studying (*).

<u>Today</u> General theory of exact cat.

<u>Next</u> Concrete study of ——.

in rep. thy of algebras.

<u>Convention.</u>

- subcat = full & closed under isom
- Λ : ring
  $\rightsquigarrow$ $\mathrm{Mod}\,\Lambda$: the cat. of right Λ-modules
  $\mathrm{mod}\,\Lambda$: —— f.g. ——.

§1. Definition

Extrinsic def.

exact str

<u>Def</u> An exact cat $(\mathcal{E}, \underline{\mathbb{E}})$ consists of

- $\mathcal{E}$: a (additive) subcat of some abelian cat $\mathcal{A}$ which is <u>closed under extensions</u>

$\mathcal{E} \ni 0$

(i.e., $\forall\, 0 \to X \to Y \to Z \to 0$: ex in $\mathcal{A}$
$X, Z \in \mathcal{E} \Rightarrow Y \in \mathcal{E}$)

- $\mathbb{E} := \left\{ 0 \to X \to Y \to Z \to 0 : \text{ex in } \mathcal{A} \;\middle|\; \text{s.t. } X, Y, Z \in \mathcal{E} \right\}$

(often omit $\mathbb{E}$ if $\mathcal{E} \subseteq \mathcal{A}$ given)

<u>Example</u> $\mathcal{A}$: abelian cat

- $\mathcal{A}$: exact cat. $\mathbb{E} = \{$all s.e.s. in $\mathcal{A}\}$
- $\mathcal{C} \subseteq \mathcal{A}$: subcat
  $\mathcal{C}^{\perp} := \{ A \in \mathcal{A} \mid \forall C \in \mathcal{C}\ \mathrm{Ext}^1_{\mathcal{A}}(C, A) = 0 \}$
  $\subseteq \mathcal{A}$: ext-closed.

- $\{\text{torsion abelian grp}\} \subseteq \operatorname{Mod} \mathbb{Z}$
  $\{\text{torsion-free} \text{ ——— }\} \subset$
  : ext-closed

- R: CM local ring
  $\rightsquigarrow \operatorname{MCM} R \subseteq \operatorname{mod} R$: ext-closed.

- k: field
  $\mathcal{E} := \{ V \in \operatorname{mod} k \mid \dim V \neq 1 \}$
  $\subset \operatorname{mod} k$: ext-closed.
  $(\quad 0, \cancel{1}, 2, 3, \dots \quad)$

Intrinsic def

Def An exact cat $(\mathcal{E}, \mathbb{E})$ consists of

- $\mathcal{E}$: an additive cat
- $\mathbb{E}$: a class of sequence
  $\delta : 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0 \quad (X, Y, Z \in \mathcal{E})$
  s.t. $\begin{cases} (1) & i = \ker p \\ (2) & p = \operatorname{coker} i \end{cases}$

which satisfy the following axioms, where

- inflation : map i above for some $\delta \in \mathbb{E}$
- deflation : —— p ——— .
- conflation : $\delta \in \mathbb{E}$.

(E-1) $\mathbb{E}$: closed under isom.

(E0) $\mathbb{E}$ contains all split exact seq.

$$\left(0 \to X \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} X \oplus Y \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} Y \to 0 \qquad X, Y \in \mathcal{E} \right)$$

(E1) inflation : closed under composition.

$(\mathrm{E1})^{op}$ defl : ———

(E2)
$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d, "\forall (2)"'] \arrow[dr, phantom, "\exists\ \text{p.o.}\ (3)"] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 & : \forall \text{ confl } (1) \\ 0 \arrow[r] & W \arrow[r] & E \arrow[r] & Z \arrow[r] & 0 & : \text{confl } (4) \end{tikzcd}$$

$(\mathrm{E2})^{op}$ omit.

(E1): "octahedral"

$$\begin{tikzcd}
 & & 0 \arrow[d, "\text{inf.}"'] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & X \arrow[r] \arrow[d] & 0 & : \text{conf.} \\
0 \arrow[r] & A \arrow[r] & C \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 & : \text{conf.} \\
 & & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & \\
 & & 0 & 0 & & \\
 & & \because\ \text{conf.} & \because\ \text{conf.} & &
\end{tikzcd}$$

Thm (Gabriel–Quillen's embedding thm)

- "Extrinsic" exact cat are intrinsic exact cat.
- Conversely any skeletally small exact cat arises extrinsically.

(i.e., $\forall\ (\mathcal{E}, \mathbb{E})$: exact cat. $\exists\ \mathcal{E} \overset{\exists}{\hookrightarrow} \mathcal{A}$: abelian cat)

In this talk,

exact cat $=$ ext-closed sub of $\exists$ abelian cat

$\dagger$ $\mathbb{E}$: a class of s.e.s. in $\mathcal{E}$ // conflation.

§2. Invariants & properties

In the rest, $(\mathcal{E}, \mathbb{E})$: exact cat.

Def

(1) $P \in \mathcal{E}$ is projective

if $\forall\ 0 \to X \to Y \to Z \to 0$: confl,

$\forall\ P \to Z$ lifts to $P \to Y$

(2) Injective obj are defined dually.

(3) $S \in \mathcal{E}$ is simple if

$S \not\cong 0$ and

$\forall\ 0 \to X \to S \to Z \to 0$: confl.

$\Rightarrow X \cong 0$ or $Z \cong 0$

$\mathrm{sim}\,\mathcal{E} := \{\text{simple objs in } \mathcal{E}\}/\cong$

Ex $\mathcal{E} = \mathrm{Mod}\,\Lambda$, then

proj, inj, simple : usual ones.

Prop For $P \in \mathcal{E}$, TFAE

(1) P : proj in $\mathcal{E}$

(2) $\forall\ 0 \to X \to Y \to P \to 0$: confl splits.

$\lrcorner$

☺ (1) ⇒ (2)

$$
\begin{tikzcd}
 & & & P \arrow[dl, dashed, "\exists"'] \arrow[d, "\mathrm{id}_P"] & \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & P \arrow[r] & 0
\end{tikzcd}
$$

(2) ⇒ (1) By $(E2)^{op}$,

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & P \arrow[r] \arrow[d, "\forall"] \arrow[l, bend left] & 0 : \text{confl.} \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 : \text{confl}
\end{tikzcd}
$$

split ∵

$\square$

Def

$\mathcal{E}$ has enough proj

$\Longleftrightarrow\ \forall\ Z \in \mathcal{E}$

$\exists\ 0 \to X \to P \to Z \to 0$: confl

with P : proj in $\mathcal{E}$.

Problem

(1) Determine proj, inj, simple objs for a given exact cat.

(2) Check whether $\mathcal{E}$ has enough proj.

$\lrcorner$

Ex

$\mathcal{E} = \{ V \in \mathrm{mod}\,k \mid \dim V \neq 1 \}$

- All objs are proj & inj (since every confl splits!)
- $\mathrm{sim}\,\mathcal{E} = \{ k^2, k^3 \}$ (by $k \notin \mathcal{E}$)

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Jordan-Hölder Property (JHP)

Def • A composition series of $X \in \mathcal{E}$ is a chain of inflations

$$0 = X_0 \rightarrowtail X_1 \rightarrowtail X_2 \cdots \rightarrowtail X_n = X$$

s.t. $X_i / X_{i-1} \in \mathrm{sim}\,\mathcal{E}$.

• $\mathcal{E}$: length

$:\Longleftrightarrow$ $\forall$ obj has a comp. ser. in $\mathcal{E}$.

• $\mathcal{E}$ satisfies (JHP)

$:\Longleftrightarrow$ $\forall$ obj $X \in \mathcal{E}$

$\mathcal{E}$: length and

$\forall$ two comp. ser. of X are equivalent

$$\left(\begin{cases} 0 = X_0 \rightarrowtail \cdots \rightarrowtail X_m = X \\ 0 = X_0' \rightarrowtail \cdots \rightarrowtail X_n' = X \end{cases}\right.$$

are equiv

$:\Longleftrightarrow$ $m = n$ and

• $X_i / X_{i-1} \cong X'_{\sigma(i)} / X'_{\sigma(i)-1}$

for some σ : perm. on $\{1, \cdots, n\}$ ∎

Ex Λ : artinian ring

$\mathrm{mod}\,\Lambda$: (JHP)

by classical JH theorem.

• $\mathcal{E} = \{ V \in \mathrm{mod}\,k \mid \dim V \neq 1 \}$

$\Rightarrow$ $\mathcal{E}$: length, but

$\mathcal{E}$: not (JHP)

$0 \subsetneq k^2 \subsetneq k^4 \subsetneq k^6$

by

~~$0 \subset k \subset \cdots \subset k^5 \subset k^6$~~

$k^6 = k^2 \oplus k^2 \oplus k^2$

$= k^3 \oplus k^3$

$0 \subset k^2 \subset k^4 \subset k^6$

$0 \subsetneq k^3 \subsetneq k^6$

$M(\mathcal{E}) = \{ n \in \mathbb{Z} \mid n \geq 0, n \neq 1 \}$

Problem Determine whether (JHP) holds for a given $\mathcal{E}$ ∎

Grothendieck grp, monoid

Def Groth. grp $K_0(\mathcal{E})$ is abelian grp defined by

$$\begin{cases} \text{generators} : [X] \text{ for } X \in \mathcal{E} \\ \text{relations} : [Y] = [X] + [Z] \end{cases}$$

for $\forall\, 0 \to X \to Y \to Z \to 0$: confl. ∎

Def Groth. monoid $M(\mathcal{E})$ is a commutative monoid defined by

gen : $[X]$ for $X \in \mathcal{E}$

rel : $\begin{cases} \cdot\, [Y] = [X] + [Z] \quad \forall\, 0 \to X \to Y \to Z \to 0 : \text{conf} \\ \cdot\, [0] = 0 \end{cases}$

Rem

$K_0(\mathcal{E})$ is obtained from $M(\mathcal{E})$ by "universal grp"

$\therefore$ $M(\mathcal{E})$ has more info than $K_0(\mathcal{E})$

Prop $\exists$ bij

$$\operatorname{sim} \mathcal{E} \xrightarrow{\sim} \{\underline{\text{atoms}} \text{ in } M(\mathcal{E})\}$$

$$X \longmapsto [X]$$

a : atom $\Leftrightarrow \begin{cases} a \neq 0 \\ a = b + c \Rightarrow b = 0 \text{ or } c = 0 \end{cases}$

$M(\mathcal{E})$ remembers simples, but $K_0(\mathcal{E})$ doesn't !

Thm TFAE

(1) $\mathcal{E}$: (JHP)

(2) $M(\mathcal{E})$ is a free monoid

i.e., $M(\mathcal{E}) \cong \bigoplus_{\exists I} \mathbb{N}$

(3) $\mathcal{E}$: length and $\{[S] \mid S \in \operatorname{sim} \mathcal{E}\}$: linearly independent in $K_0(\mathcal{E})$

In this case, $M(\mathcal{E})$: free with basis $\{[S] \mid S \in \operatorname{sim} \mathcal{E}\}$

(Sketch)

(1) $\Rightarrow$ (2) :

$$\underbrace{\bigoplus_{S \in \operatorname{sim} \mathcal{E}} \mathbb{N} \cdot [S]}_{\text{free monoid}} \longrightarrow M(\mathcal{E}) \quad : \text{natural map}$$

is surj

by $\mathcal{E}$: length

$0 \rightarrowtail X_1 \rightarrowtail \cdots \rightarrowtail X_n = X$: comp ser

$\Rightarrow$ $[X] = \underset{\sim}{[X_1]} + [X_2/X_1] + \cdots + [X_n/X_{n-1}]$

X_i/X_{i-1} $\Uparrow$ sim $\mathcal{E}$ in $M(\mathcal{E})$

- $M(\mathcal{E}) \longrightarrow \bigoplus \mathbb{N} \cdot [S]$

$\quad \cup$

$\quad [X] \longmapsto \displaystyle\sum_{\substack{0 \to \cdots \to X \\ \text{: comp. ser.}}} [X_i/X_{i-1}]$

: well-defined by

$\mathcal{E}$: (JHP)

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