

# Exact-categorical properties of subcategories of abelian categories

(Part I: General theory)

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§0. Intro

§1. Definitions

§2. Invariants and properties of exact cat

§0.

Exact cat: introduced by Quillen in 1973 [Higher algebraic K-theory]

Exact cat  $\mathcal{E}$   
= additive cat + extra str  $\mathbb{E}$   
( $\mathbb{E}$ : class of "short exact seq" in  $\mathcal{E}$ )

• Extension-closed subcat of an abelian cat has a natural exact str.

(but exact str are not uniquely determined by  $\mathcal{E}$ )

Q. Why Exact Category?

A0. To define higher algebraic K-grp.

A1. It provides a framework for doing homological alg for (not necessarily abelian) additive cat. (e.g. Banach sp, ...)

A2. Particular exact cat (Frobenius) gives a triangulated cat. (called "algebraic tri. cat") (e.g. homotopy cat of abelian cat)

My answer.

(\*)

To study subcat of an abelian cat!

In fact, by regarding (\*) as exact cat, we can consider properties and invariants of (\*)

This gives us a lot of strategy & problem when studying (\*).

Today General theory of exact cat.

Next Concrete study of \_\_\_\_\_ in rep. thy of algebras.

Convention.

• subcat = full & closed under isom

•  $\Lambda$  : ring

$\rightsquigarrow \text{Mod } \Lambda$  : the cat. of right  $\Lambda$ -modules  
mod  $\Lambda$  : \_\_\_\_\_ f.g. \_\_\_\_\_

§1. Definition

exact str

Extrinsic def.

↓

Def An exact cat  $(\mathcal{E}, \underline{\mathbb{E}})$

consists of (additive)

•  $\mathcal{E}$  : a subcat of some abelian  $\mathcal{A}$  which is closed under extensions

(i.e.,  $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 : \text{ex in } \mathcal{A}$   
 $X, Z \in \mathcal{E} \Rightarrow Y \in \mathcal{E}$ )

•  $\mathbb{E} := \left\{ \begin{array}{l} 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 : \text{ex in } \mathcal{A} \\ \text{s.t. } X, Y, Z \in \mathcal{E} \end{array} \right\}$

(often omit  $\mathbb{E}$  if  $\mathcal{E} \subseteq \mathcal{A}$  given)

Example  $\mathcal{A}$  : abelian cat

•  $\mathcal{A}$  : exact cat.  $\mathbb{E} = \left\{ \begin{array}{l} \text{all s.p.s. in } \mathcal{A} \end{array} \right\}$

•  $\mathcal{E} \subseteq \mathcal{A}$  : subcat

$\mathcal{E}^\perp := \left\{ A \in \mathcal{A} \mid \forall C \in \mathcal{E} \quad \mathcal{A}(C, A) = 0 \right\}$   
 $\subseteq \mathcal{A}$  : ext-closed.

$\circ \{ \text{torsion abelian grp} \} \subseteq \text{Mod } \mathbb{Z}$   
 $\left\{ \begin{array}{l} \text{torsion-free} \\ \text{---} \end{array} \right\} C$   
 $: \text{ext-closed}$

$\circ R: \text{ CM local ring}$   
 $\rightarrow \text{MCM } R \subseteq \text{mod } R: \text{ ext-closed.}$

$\circ k: \text{ field}$   
 $E := \{ V \in \text{mod } k \mid \dim V \neq 1 \}$   
 $C \text{ mod } k: \text{ ext-closed.}$   
 $(0, X, 2, 3, \dots)$

### Intrinsic def

Def An exact cat  $(\mathcal{E}, \mathbb{E})$  consists of

- $\circ \mathcal{E}$ : an additive cat
  - $\circ \mathbb{E}$ : a class of sequence
- $$\delta: 0 \rightarrow X \xrightarrow{i} Y \xrightarrow{p} Z \rightarrow 0 \quad (X, Y, Z) \in \mathcal{E}$$
- $\text{s.t. } \begin{cases} (1) & i = \ker p \\ (2) & p = \text{coker } i \end{cases}$

which satisfy the following axioms,  
where

- $\circ$  inflation:  $\text{map } i$  above  
for some  $\delta \in \mathbb{E}$
- $\circ$  deflation:  $\text{--- } p \text{ ---}$
- $\circ$  conflation:  $\delta \in \mathbb{E}$ .

(E-1)  $\mathbb{E}$ : closed under isom.

(E=0)  $\mathbb{E}$  contains all split exact seq.

$$\left( \begin{array}{ccccccc} 0 & \rightarrow & X & \rightarrow & X \oplus Y & \rightarrow & Y & \rightarrow & 0 \\ & & & & \downarrow [0] & & \downarrow [0] & & \\ & & & & & & & & X, Y \in \mathcal{E} \end{array} \right)$$

(E1) inflation: closed under composition.

(E1)<sup>op</sup> defl:

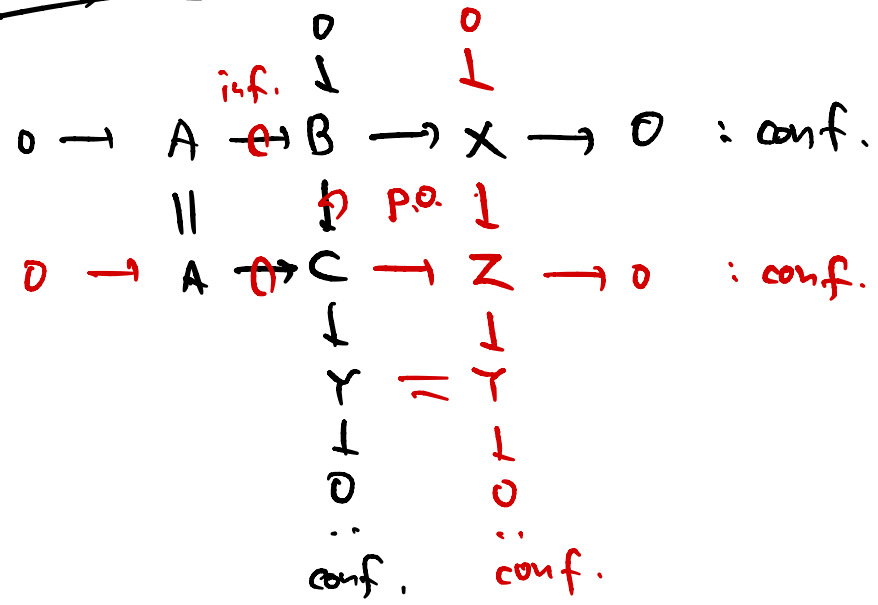
(E2)  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0: \forall \text{ conf l } \textcircled{1}$

$$\forall \textcircled{2} \downarrow \begin{matrix} \mathbb{E} \\ \text{P.O.} \\ \textcircled{3} \end{matrix} \downarrow \parallel$$

$0 \rightarrow W \rightarrow E \rightarrow Z \rightarrow 0: \text{ conf l } \textcircled{4}$

(E2)<sup>op</sup> omit.

(E1): "octahedral"



Thm (Gabriel-Quillen's embedding thm)  
 "Extrinsic" exact cat are  
 intrinsic exact cat.

• Conversely any skeletally small  
 exact cat arises extrinsically.

(i.e.,  $\forall (E, \mathbb{E})$ : exact cat.  
 $\exists E \hookrightarrow \mathcal{A}$ : abelian cat)

In this talk,  
 exact cat = ext.-closed sub of  
 $\exists$  abelian cat  
 $\mathbb{E}$ : a class of s.e.s. in E

§2. Invariants & properties

In the rest,  $(E, \mathbb{E})$ : exact cat.

Def

(1)  $P \in E$  is projective

if  $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  : conf.

$\forall P \rightarrow Z$  lifts to  $P \rightarrow Y$

(2) Injective obj are defined dually.

(3)  $S \in E$  is simple if

$S \neq 0$  and

$\forall 0 \rightarrow X \rightarrow S \rightarrow Z \rightarrow 0$  : conf.

$\rightarrow X \cong 0$  or  $Z \cong 0$

$\text{sim } E := \{ \text{simple objs in } E \} / \cong$

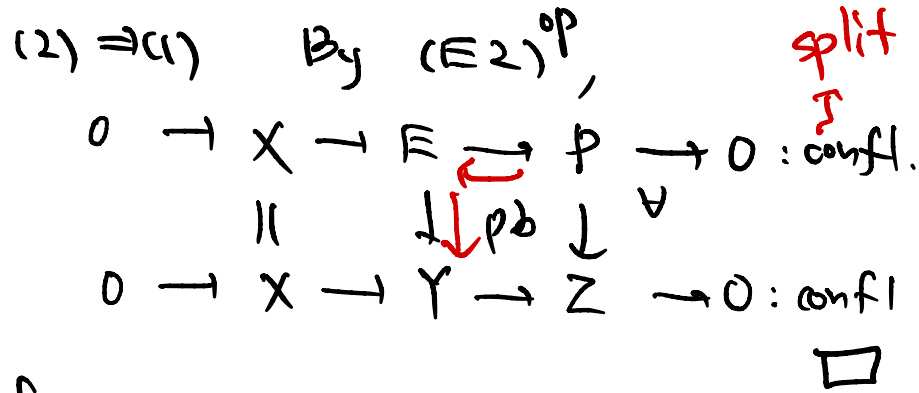
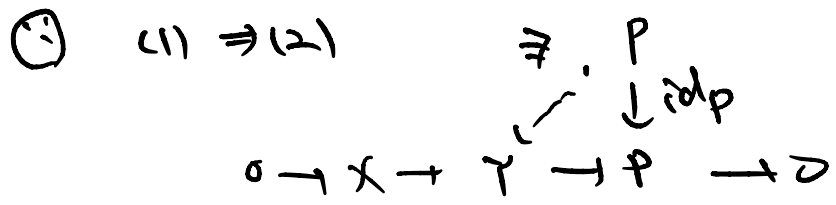
Ex  $E = \text{Mod } \Lambda$ , then

proj, inj, simple :  
 usual ones.

conflation.

Prop For  $P \in \mathcal{E}$ , TFAE

- (1)  $P$  : proj in  $\mathcal{E}$
- (2)  $\forall 0 \rightarrow X \rightarrow Y \rightarrow P \rightarrow 0$  : confl splits.



Def  
 $\mathcal{E}$  has enough proj

$\Leftrightarrow \forall Z \in \mathcal{E}$   
 $\exists 0 \rightarrow X \rightarrow P \rightarrow Z \rightarrow 0$  : confl  
 with  $P$  : proj in  $\mathcal{E}$ .

Problem

- (1) Determine proj, inj, simple objs for a given exact cat.
- (2) Check whether  $\mathcal{E}$  has enough proj.

Ex

$$\mathcal{E} = \{ V \in \text{mod } k \mid \dim V \neq 1 \}$$

- All objs are proj & inj (since every confl splits!)
- $\text{sim } \mathcal{E} = \{ k^2, k^3 \}$  (by  $k \notin \mathcal{E}$ )

Jordan-Hölder Property (JHP)

Def • A composition series of  $X \in \mathcal{E}$  is a chain of inflations  
 $0 = X_0 \rightarrow X_1 \rightarrow X_2 \dots \rightarrow X_n = X$   
 s.t.  $X_i / X_{i-1} \in \text{sim } \mathcal{E}$ .

•  $\mathcal{E}$ : length

$\Leftrightarrow \forall \text{ obj has a comp. ser. in } \mathcal{E}$ .

•  $\mathcal{E}$  satisfies (JHP)

$\Leftrightarrow \forall \text{ obj } X \in \mathcal{E}$

$\forall$  two comp. ser. of  $X$  are equivalent

$\mathcal{E}$ : length and

$$\left( \begin{array}{l} 0 = x_0 \rightarrow \dots \rightarrow x_m = X \\ 0 = x'_0 \rightarrow \dots \rightarrow x'_m = X \end{array} \right.$$

are equiv

$\Leftrightarrow m = n$  and

$$x_i / x_{i-1} \cong x'_{\sigma(i)} / x'_{\sigma(i)-1}$$

for some  $\sigma$ : perm. on  $\{1, \dots, n\}$

Ex

$\Lambda$ : artinian ring

$\text{mod } \Lambda$ : (JHP)

by classical JH theorem.

•  $\mathcal{E} = \{v \in \text{mod } k \mid \dim v \neq 1\}$

$\neq \mathcal{E}$ : length, but

$\mathcal{E}$ : not (JHP)

$$0 \subset k^2 \subset k^4 \subset k^6$$

by

$$0 \subset k^2 \subset k^4 \subset k^6 = k^2 \oplus k^2 \oplus k^2$$

$$0 \subset k^2 \subset k^4 \subset k^6 = k^3 \oplus k^3$$

$$0 \subset k^2 \subset k^4 \subset k^6 \quad \downarrow \quad 0 \subset k^3 \subset k^6$$

Problem

$$M(\mathcal{E}) = \{n \in \mathbb{Z} \mid n \geq 0, n \neq 1\}$$

Determine whether (JHP) holds for a given  $\mathcal{E}$

Grothendieck grp, monoid

Def Groth. grp  $K_0(\mathcal{E})$  is abelian grp defined by

generators:  $[X]$  for  $X \in \mathcal{E}$

relations:  $[Y] = [X] + [Z]$

for  $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ : confl.

Def Groth. monoid  $M(\mathcal{E})$  is a commutative monoid defined by

gen :  $[x]$  for  $x \in \mathcal{E}$

rel :  $\begin{cases} \bullet [Y] = [X] + [Z] \\ \quad \forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 : \text{conf} \\ \bullet [0] = 0 \end{cases}$

Rem

$K_0(\mathcal{E})$  is obtained from  $M(\mathcal{E})$

by "universal grp"

$\therefore M(\mathcal{E})$  has more info than  $K_0(\mathcal{E})$  *a: atom*

Prop

$\exists$  bij

$\left\{ \begin{array}{l} a \neq 0 \\ a = b + c \\ \Rightarrow b = 0 \text{ or } c = 0 \end{array} \right.$

$\text{Sim } \mathcal{E} \cong \{ \text{atoms in } M(\mathcal{E}) \}$   
 $X \mapsto [X]$

$M(\mathcal{E})$  remembers simples, but  $K_0(\mathcal{E})$  doesn't!

Thm TFAE

(1)  $\mathcal{E} : (\text{JHP})$

(2)  $M(\mathcal{E})$  is a free monoid

i.e.,  $M(\mathcal{E}) \cong \bigoplus_{\mathbb{N}} \mathbb{N}$

(3)  $\mathcal{E} : \text{length}$  and

$\{ [S] \mid S \in \text{sim } \mathcal{E} \}$  :

linearly independent in  $K_0(\mathcal{E})$

In this case,

$M(\mathcal{E})$  : free with

basis  $\{ [S] \mid S \in \text{sim } \mathcal{E} \}$

(Sketch)

(1)  $\Rightarrow$  (2) :

$\bigoplus_{S \in \text{sim } \mathcal{E}} \mathbb{N} \cdot [S]$   
 free monoid

$\longrightarrow M(\mathcal{E})$  : natural map

is surj

by  $\varepsilon$ : length

$$0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n = X : \text{comp ser}$$

$$\Rightarrow [X] = \underbrace{[x_1]}_{\substack{x_1/x_0 \\ \uparrow \\ \text{sim } \varepsilon}} + [x_2/x_1] + \dots + [x_n/x_{n-1}] \in M(\varepsilon)$$

$$\bullet M(\varepsilon) \rightarrow \bigoplus \mathbb{N} \cdot [S]$$

$$\cup \\ [X] \mapsto \sum_{0 \rightarrow \dots \rightarrow X : \text{comp. ser.}} [x_i/x_{i-1}]$$

: well-defined by

$$\varepsilon : (\text{JHP})$$

$\frac{1}{21}$

$\frac{1}{21}$

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