Cyclic polytopes and higher Auslander–Reiten theory

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Overview

Algebra	Combinatorics	
Clusters in A_n cluster algebra	Δations of convex polygons	[FZ02; FZ03]
Cluster-tilting objects for A_n^d	Δations of $2d$-dim cyclic polytopes	[OT12]
Riedtmann–Schofield orders	Higher Stasheff–Tamari orders	[BK04, $d = 1$], [Wil21a]
d-maximal green sequences of A_n^d	Δations of $(2d + 1)$-dim cyclic polytopes	[Wil21a]
Orders on d-maximal green sequences	Higher Stasheff–Tamari orders	[Wil21a]

Plan

1. Cyclic polytopes
2. Triangulations of cyclic polytopes
3. The higher Stasheff–Tamari orders
4. Higher Auslander–Reiten theory
5. Even-dimensional cyclic polytopes in representation theory

Plan

6. The HST orders in HAR theory

7. Equality of the higher Stasheff–Tamari orders

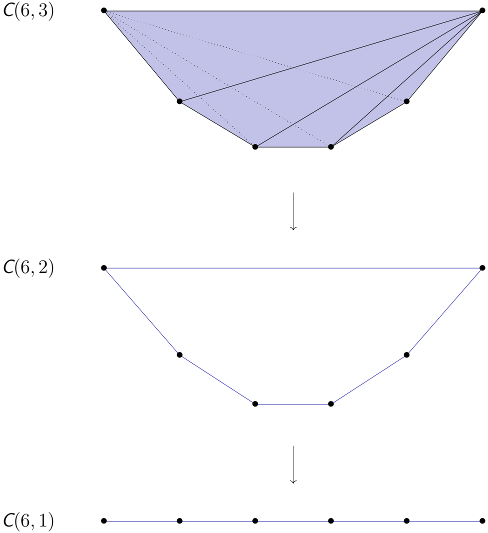
8. Mutation

1. Cyclic polytopes

Cyclic polytopes

The *cyclic polytope* $C(m, \delta)$ is the convex hull of m points $\{p_{t_1}, \ldots, p_{t_m}\} \subset \mathbb{R}^\delta$ on the *moment curve* $p_t = (t, t^2, \ldots, t^\delta)$, where $\{t_1, \ldots, t_m\} \subset \mathbb{R}$.

We have projections $C(m, \delta + 1) \to C(m, \delta)$ given by forgetting the last coordinate.



[ER96, Figure 2]

Combinatorics of cyclic polytopes: facets

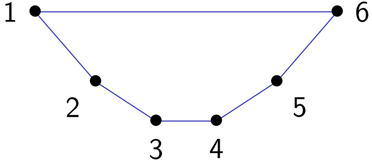
Recall that a *facet* of a polytope is a face of codimension one.

The *upper (lower) facets* of the cyclic polytope $C(m, \delta)$ are those that can be seen from points with a very large positive (negative) δ-th coordinate.

Given $F \subset [m]$ where $\#F = \delta$, then $|F|$ is an upper (lower) facet of $C(m, \delta)$ if and only if for all $i \in [m] \setminus F$,

$$\#\{j \in F : j > i\}$$

is odd (even). (*Gale's Evenness Criterion*.)



Upper $= \{16\}$

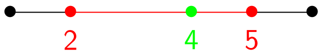
Lower $= \{12, 23, 34, 45, 56\}$

Combinatorics of cyclic polytopes: circuits

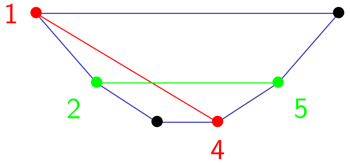
A *circuit* of a polytope is a pair (A, B) of disjoint sets of vertices such that $\mathrm{conv}(A) \cap \mathrm{conv}(B) \neq \varnothing$ such that A and B are minimal with respect to this property.

The circuits of $C(m, \delta)$ are the pairs $(Z_-, Z_+), (Z_+, Z_-)$ where $Z_- = \{\dots, z_{\delta-1}, z_{\delta+1}\}, Z_+ = \{\dots, z_\delta, z_{\delta+2}\}$ for $\{z_1, z_2, \dots, z_{\delta+1}, z_{\delta+2}\} \subseteq [m]$.

Here we say that Z_- *intertwines* Z_+ and write $Z_- \wr Z_+$.



(25,4), (4,25)

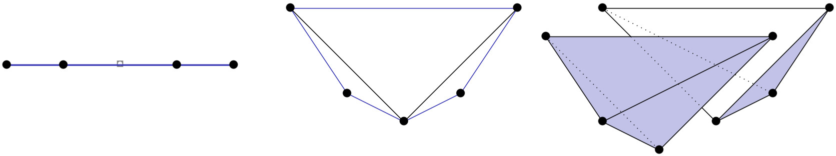


(14,25), (25,14)

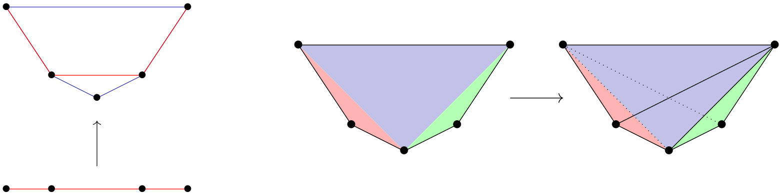
2. Triangulations of cyclic polytopes

Triangulations and sections

A *triangulation* of $C(m, \delta)$ is a subdivision of $C(m, \delta)$ into δ-simplices whose vertices are vertices of $C(m, \delta)$.



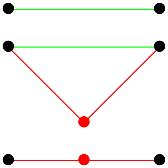
Triangulations $\mathcal{T}$ give sections $s_{\mathcal{T}} \colon C(m, \delta) \to C(m, \delta + 1)$. These are composed of simplex-wise maps $s_A \colon |A| \to C(m, \delta + 1)$ for simplices $|A|$.

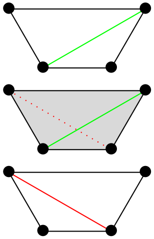


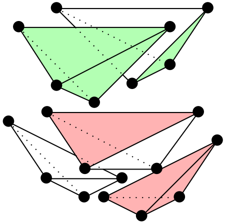
Bistellar flips

Given a $(\delta+1)$-simplex $|S|$ on the moment curve in $\mathbb{R}^{\delta+1}$, both the upper facets and the lower facets of $|S|$ project to a triangulation of $C(\delta+2,\delta)$.

An *increasing bistellar flip* on a triangulation $\mathcal{T}$ of $C(m,\delta)$ consists of replacing a triangulation of a $C(\delta+2,\delta)$ subpolytope coming from the lower facets of some $(\delta+1)$-simplex $|S|$ with the triangulation coming from the upper facets.







Description of even-dimensional triangulations

A triangulation of a convex polygon is given by a set of non-crossing arcs of a particular size.

A similar description holds for even-dimensional cyclic polytopes.

Theorem ([OT12])

A triangulation of $C(m, 2d)$ is given by a set of size $\binom{m-d-2}{d}$ of internal d-simplices which do not intersect transversely.

Combinatorial description for even dimensions

Theorem ([OT12])

There is a bijection between triangulations of $C(m, 2d)$ and sets of non-intertwining $(d+1)$-tuples from ${}^{\circlearrowleft}\mathbf{I}_m^d$ of size $\binom{m-d-2}{d}$.

A d-simplex $|A|$ in $C(m, 2d)$ is internal if and only if

$$A \in {}^{\circlearrowleft}\mathbf{I}_m^d := \left\{ (a_0, \ldots, a_d) \in [m]^{d+1} \mid a_{i+1} \geqslant a_i + 2 \mod m \right\}.$$

We know from the description of the circuits of $C(m, 2d)$ when a pair of d-simplices $|A|$ and $|B|$ intersect transversely, namely when $A \wr B$ or $B \wr A$.

3. The higher Stasheff–Tamari orders

The first higher Stasheff–Tamari order $\mathcal{S}_1(m, \delta)$

Defined first by Kapranov and Voevodsky and then by Edelman and Reiner in a different way. Thomas showed the two definitions gave the same order.

We have that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathcal{T}'$ is the result of performing an increasing bistellar flip within $\mathcal{T}$.

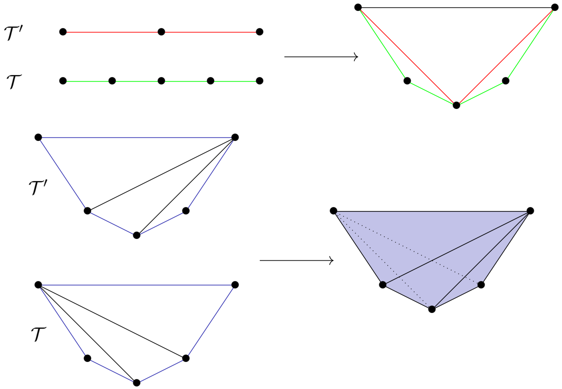
Hence $\mathcal{T} <_1 \mathcal{T}'$ if and only if we have

$$\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'.$$

The second higher Stasheff–Tamari order $\mathcal{S}_2(m, \delta)$

Defined by Edelman and Reiner [ER96]. Given $\mathcal{T}, \mathcal{T}'$ triangulations of $C(m, \delta)$,

$$\mathcal{T} \leqslant_2 \mathcal{T}' \iff s_{\mathcal{T}}(x)_{\delta+1} \leqslant s_{\mathcal{T}'}(x)_{\delta+1} \quad \forall x \in C(m, \delta).$$



Submersion sets

Edelman and Reiner give the following alternative characterisation of the second higher Stasheff–Tamari order.

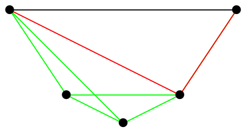
Given a simplex $|A|$ in $C(m,\delta)$, recall the map $s_A \colon |A| \to C(m,\delta+1)$.

A simplex $|A|$ is *submerged* by a triangulation $\mathcal{T}$ if

$$s_A(x)_{\delta+1} \leqslant s_{\mathcal{T}}(x)_{\delta+1} \quad \forall x \in |A|.$$

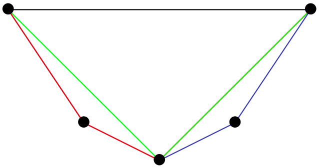
The *k-submersion set* $\mathrm{sub}_k \mathcal{T}$ is the set of k-simplices submerged by the triangulation $\mathcal{T}$.

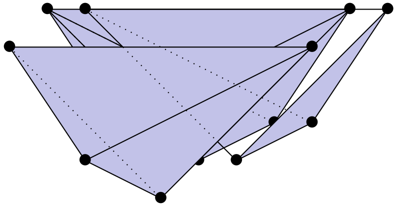
Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\mathrm{sub}_{\lceil \delta/2 \rceil} \mathcal{T} \subseteq \mathrm{sub}_{\lceil \delta/2 \rceil} \mathcal{T}'$.



Rambau's Theorem

$$\left\{\begin{matrix}\text{Triangulations of} \\ C(m, \delta+1)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Maximal chains in} \\ \mathcal{S}_1(m, \delta)\end{matrix}\right\} \Big/ \sim$$





The equality of the higher Stasheff–Tamari orders

Edelman and Reiner conjectured that $\mathcal{S}_1(m,\delta) = \mathcal{S}_2(m,\delta)$ in their 1996 paper, which they proved for $\delta \leqslant 3$.

In [Wil21b], we prove that their conjecture is true for all δ. We will briefly sketch the proof later in the talk.

It is clear that if $\mathcal{T} \leqslant_1 \mathcal{T}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$, since an increasing bistellar flip moves the section upwards.

But it is not clear that we always have $\mathcal{T} \leqslant_1 \mathcal{T}'$ whenever $\mathcal{T} \leqslant_2 \mathcal{T}'$, since it is not obvious how to construct a sequence of increasing bistellar flips from $\mathcal{T}$ to $\mathcal{T}'$.

(Lack of) lattice property of the higher Stasheff–Tamari orders

Edelman and Reiner also showed that $\mathcal{S}_1(m, \delta)$ and $\mathcal{S}_2(m, \delta)$ are lattices for $\delta \leqslant 3$.

Edelman, Rambau, and Reiner found a counter-example to $\mathcal{S}_2(m, \delta)$ always being a lattice.

The same counter-example was used to show that $\mathcal{S}_1(m, \delta)$ is not always a lattice in [Wil21a].

4. Higher Auslander–Reiten theory

Higher Auslander–Reiten theory

Introduced by Iyama as a higher-dimensional generalisation of classical Auslander–Reiten theory.

Given a finite-dimensional K-algebra Λ over a field K, a functorially finite subcategory $\mathcal{M}$ of $\operatorname{mod}\Lambda$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{M} &= \{X \in \operatorname{mod}\Lambda \mid \forall M \in \mathcal{M}, \operatorname{Ext}_{\Lambda}^{1,\dots,d-1}(X, M) = 0\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \forall M \in \mathcal{M}, \operatorname{Ext}_{\Lambda}^{1,\dots,d-1}(M, X) = 0\}.\end{aligned}$$

If $\operatorname{add} M$ is a d-cluster-tilting subcategory, then M is called a *d-cluster-tilting module*.

If Λ has a d-cluster-tilting module M and $\operatorname{gl.dim}\Lambda \leqslant d$, then Λ is called *d-representation-finite d-hereditary*, following [IO11; HIO14]. In this case, $\operatorname{add} M$ is unique.

Higher quivers of type A

Following [OT12],

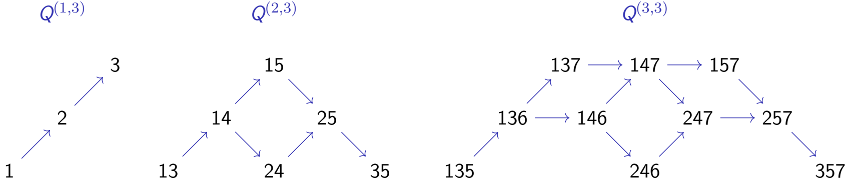
$$\mathbf{I}_m^d := \{(a_0, \ldots, a_d) \in [m]^{d+1} \mid \forall i \in \{0, 1, \ldots, d-1\}, a_{i+1} \geqslant a_i + 2\}$$

Let $Q^{(d,n)}$ be the quiver with vertices $Q_0^{(d,n)} := \mathbf{I}_{n+2d-2}^{d-1}$ and arrows

$$Q_1^{(d,n)} := \{A \to A + 1_i \mid A, A + 1_i \in Q_0^{(d,n)}\},$$

where

$$1_i := (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0).$$



Higher Auslander algebras of type A

Let A_n^d be the quotient of the path algebra $KQ^{(d,n)}$ by the relations:

$$A \to A + 1_i \to A + 1_i + 1_j = \begin{cases} A \to A + 1_j \to A + 1_i + 1_j & A + 1_j \in Q_0 \\ 0 & \text{otherwise.} \end{cases}$$

We multiply arrows as if we were composing functions, so that $\xrightarrow{\alpha}\xrightarrow{\beta} = \beta\alpha$.

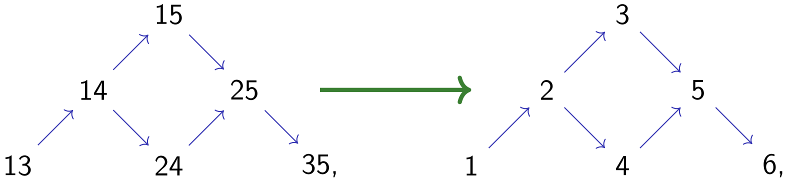
Theorem ([Iya11])

A_n^d is d-representation-finite d-hereditary with unique basic d-cluster-tilting module $M^{(d,n)}$ and

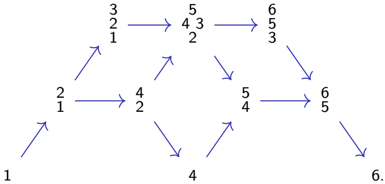
$$\mathrm{End}_{A_n^d} M^{(d,n)} \cong A_n^{d+1}.$$

The d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$: example

If we label $Q^{(2,3)}$ as



then the 2-cluster-tilting subcategory of $\operatorname{mod} A_3^2$ is given by



It can be seen that this is the same as the quiver $Q^{(3,3)}$.

The d-almost positive category

Given a d-representation-finite d-hereditary algebra Λ with d-cluster-tilting module M, define the *d-almost positive* category $\mathcal{U}_{\Lambda}^{\{-d,0\}}$ to be the subcategory $\operatorname{add}(M \oplus \Lambda[d])$ of $D^b(\operatorname{mod} A_n^d)$.

For $d = 1$, this coincides with the category of two-term complexes of projectives.

But, for $d > 1$, this category does not contain all $(d+1)$-term complexes of projectives.

The d-AP category for type A

Theorem ([Wil; OT12])

There is a bijection $A \mapsto U_A$ between ${}^{\circlearrowleft}\mathbf{I}^{d}_{n+2d+1}$ and the indecomposable objects of $\mathcal{U}^{\{-d,0\}}_{A^d_n}$ such that:

- $\mathrm{Hom}_{D^b(\mathrm{mod}\, A^d_n)}(U_A, U_B[d]) \neq 0$ *if and only if $B \wr A$, and in this case the* Hom*-space is one-dimensional.*

We will use this combinatorial interpretation of the d-almost positive category of type A to make the algebraic connection with cyclic polytopes.

Other properties of the categories $\mathcal{U}^{\{-d,0\}}_{A^d_n}$ are also encoded combinatorially.

The d-AP category for type A: example

If we label $Q^{(2,2)}$ by

$$\begin{array}{ccccc} & 14 & & & \\ \nearrow & & \searrow & & \\ 13 & & 24 \end{array} \quad \longrightarrow \quad \begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & & 3. \end{array}$$

the 2-almost positive category of A_2^2 is

$$\begin{array}{ccccccccccc} & & \begin{smallmatrix}2\\1\end{smallmatrix} & \longrightarrow & \begin{smallmatrix}3\\2\end{smallmatrix} & & & & 1[2] & \longrightarrow & \begin{smallmatrix}2\\1\end{smallmatrix}[2] & & \\ & \nearrow & & & & \searrow & & \nearrow & & & & \searrow & \\ 1 & & & & & & 3 & & & & & & \begin{smallmatrix}3\\2\end{smallmatrix}[2] \end{array}$$

$$\begin{array}{ccccccccccccc} & & U_{136} & \longrightarrow & U_{146} & & & & U_{247} & \longrightarrow & U_{257} & & \\ & \nearrow & & & & \searrow & & \nearrow & & & & \searrow & \\ U_{135} & & & & & & U_{246} & & & & & & U_{357} \end{array}$$

5. Even-dimensional cyclic polytopes in representation theory

Silting complexes

A complex T in $D^b(\mathrm{mod}\,\Lambda)$ is called *pre-silting* if $\mathrm{Hom}_{D^b(\mathrm{mod}\,\Lambda)}(T, T[i]) = 0$ for all $i > 0$.

A pre-silting complex T in $D^b(\mathrm{mod}\,\Lambda)$ is called *silting* if, additionally, $\mathrm{thick}\, T = D^b(\mathrm{mod}\,\Lambda)$.

Here $\mathrm{thick}\, T$ denotes the smallest full subcategory of $D^b(\mathrm{mod}\,\Lambda)$ which contains T and is closed under cones, $[\pm 1]$, direct summands, and isomorphisms.

d-silting complexes

We call a silting object T of $D^b(\operatorname{mod}\Lambda)$ *d-silting* if, additionally, it lies in $\mathcal{U}_\Lambda^{\{-d,0\}}$.

Note that for objects T, T' of $\mathcal{U}_\Lambda^{\{-d,0\}}$ we have $\operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(T, T'[i]) = 0$ if $i \notin \{-d, 0, d\}$ due to the d-cluster-tilting condition and the global dimension of Λ.

Hence, for an object T of $\mathcal{U}_\Lambda^{\{-d,0\}}$ with $\operatorname{thick} T = D^b(\operatorname{mod}\Lambda)$ to be d-silting, it suffices that $\operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(T, T[d]) = 0$.

Triangulations and d-silting complexes

Theorem ([Wil; OT12])

There are bijections between:

- ${}^{\circlearrowleft}\mathbf{I}^{d}_{n+2d+1}$,
- *internal d-simplices of* $C(n+2d+1, 2d)$,
- *isoclasses of indecomposables in* $\mathcal{U}^{\{-d,0\}}_{A^d_n}$,

which induce bijections between:

- *non-intertwining subsets of* ${}^{\circlearrowleft}\mathbf{I}^{d}_{n+2d+1}$ *of size* $\binom{n+d-1}{d}$,
- *triangulations of* $C(n+2d+1, 2d)$,
- *basic d-silting complexes in* $\mathcal{U}^{\{-d,0\}}_{A^d_n}$.

Triangulations and d-silting complexes: sketch proof

We already know the bijection between the first two items from the combinatorial description of triangulations of even-dimensional cyclic polytopes from [OT12].

Since $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(U_A, U_B[d]) \neq 0$ if and only if $B \wr A$, we have that non-intertwining sets of $(d+1)$-tuples from ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ correspond to pre-silting complexes in $\mathcal{U}^{\{-d,0\}}_{A_n^d}$.

One can then show that basic presilting complexes in $\mathcal{U}^{\{-d,0\}}_{A_n^d}$ with $\binom{n+d-1}{d}$ isoclasses of indecomposable summands are in fact silting.

6. The HST orders in HAR theory

Even dimensions

Theorem ([Wil21a])

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d+1, 2d)$ corresponding to d-silting complexes T and T' for A_n^d. We then have that

1. *$\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if T' is a left mutation of T; and*
2. *$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.*

Left mutation: $T = E \oplus X$, $T' = E \oplus Y$, where $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(Y, X[d]) \neq 0$.

$${}^{\perp}T = \{X \in \mathcal{U}_{A_n^d}^{\{-d,0\}} \mid \mathrm{Hom}(X, T[i]) = 0 \; \forall i > 0\}.$$

Even dimensions: sketch of proof

1. One can show that if $|A \cup B|$ is a $(2d+1)$-simplex with $A \wr B$ inducing a bistellar flip on a triangulation $\mathcal{T}$, then this bistellar flip replaces the internal d-simplex $|A|$ with the internal d-simplex $|B|$ to give the new triangulation $\mathcal{T}'$.

Hence, if T and T' are the corresponding d-silting complexes, then we have $T = E \oplus U_A$, $T' = E \oplus U_B$ with $A \wr B$.

Since we know that $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(U_B, U_A[d]) \neq 0$ if and only if $A \wr B$, we obtain that T' is a left mutation of T.

Even dimensions: sketch of proof

2. One can show that an internal d-simplex $|A|$ is submerged by a triangulation $\mathcal{T}$ if and only if there is no internal d-simplex $|B|$ of $\mathcal{T}$ such that $B \wr A$.

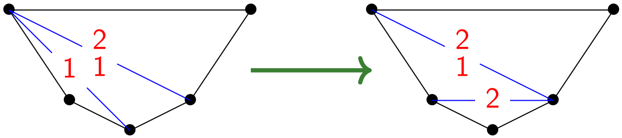
   Again, since $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_n^d)}(U_B, U_A[d]) \neq 0$ if and only if $A \wr B$, we have that $|A|$ is submerged by $\mathcal{T}$ if and only if $U_A \in {}^{\perp}T$.

   The result then follows from the interpretation of the second higher Stasheff–Tamari order in terms of submersion sets.

Illustration of even dimensions

We consider the case of $\mathcal{U}_{A_2^1}^{\{-1,0\}}$.

There is a left mutation from $T = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ to $T' = 2 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ since $\mathrm{Hom}_{D^b(\mathrm{mod}\, A_2)}(2, 1[1]) \neq 0$, corresponding to the bistellar flip



Then ${}^\perp T = \{1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\} \subseteq \{1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, 2\} = {}^\perp T'$, corresponding to the fact the second higher Stasheff–Tamari order holds.

d-maximal green sequences

We know from Rambau's theorem that triangulations of $C(n+2d+1, 2d+1)$ are given by equivalence classes of maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$.

We know from our algebraic interpretation of the higher Stasheff–Tamari orders in dimension $2d$ that maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ correspond to sequences of left mutations from A_n^d to $A_n^d[d]$ in $\mathcal{U}_{A_n^d}^{\{-d,0\}}$.

For $d = 1$, a sequence of left mutations from the projectives to the shifted projectives is a *maximal green sequence*.

Hence, we define a *d-maximal green sequence* of a d-representation-finite d-hereditary algebra Λ as a sequence of left mutations from Λ to $\Lambda[d]$ in $\mathcal{U}_{\Lambda}^{\{-d,0\}}$.

Equivalence of d-maximal green sequences

In Rambau's theorem, we have that *equivalence classes* of maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ correspond to triangulations of $C(n+2d+1, 2d+1)$.

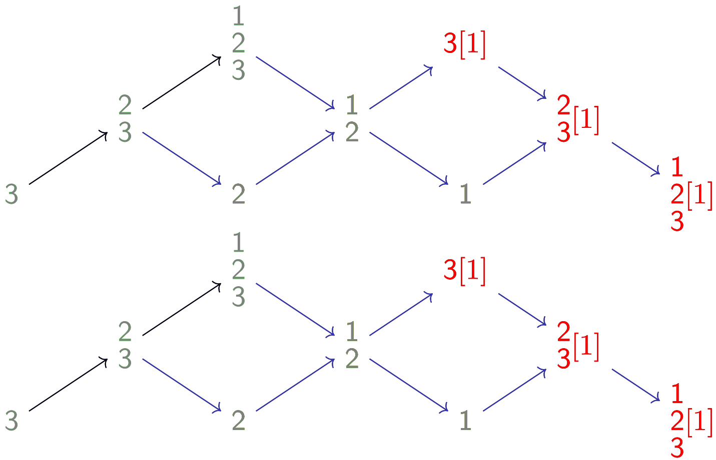
Hence, in order to get a bijection with odd-dimensional triangulations, we put an equivalence relation on d-maximal green sequences, which is as follows.

Given a d-maximal green sequence G, we write $\Sigma(G)$ for the set of indecomposable summands of objects occurring in G.

We write $G \sim G'$ if $\Sigma(G) = \Sigma(G')$ and write $\widetilde{\mathcal{MG}}_d(A_n^d)$ for the set of $\sim$-equivalence classes of d-maximal green sequences of A_n^d.

Equivalence of *d*-maximal green sequences: example

For example, for the algebra A_3, the following two maximal green sequences are equivalent:



Algebraic bijection for odd-dimensional triangulations

Theorem ([Wil21a])

There is a bijection between $\widetilde{\mathcal{MG}}_d(A_n^d)$ *and triangulations of* $C(n+2d+1, 2d+1)$.

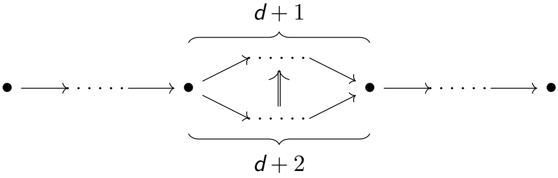
Odd dimensions

Theorem ([Wil21a])

Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G], [G'] \in \widetilde{\mathcal{MG}}_d(A_n^d)$. We then have that

1. *$\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G}' \in [G']$ such that $\widehat{G}'$ is an increasing elementary polygonal deformation of $\widehat{G}$; and*
2. *$\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

Increasing elementary polygonal deformations:



Odd dimensions: sketch of proof

1. The $(2d+2)$-simplex inducing a bistellar flip has $d+1$ $(2d+1)$-simplices as its upper facets and $d+2$ $(2d+1)$-simplices as its lower facets.

   Each of these $(2d+1)$-simplices corresponds to a bistellar flip, and so a left mutation in the d-maximal green sequence.

   We can choose equivalence-class representatives such that all these $(2d+1)$-simplices occur in a row.

   Hence, we get an increasing elementary polygonal deformation as described.

Odd dimensions: sketch of proof

1. To prove this, we show that the second higher Stasheff–Tamari order can be interpreted in terms of supermersion sets.

   Indeed, $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\sup_{\lfloor \delta/2 \rfloor} \mathcal{T} \supseteq \sup_{\lfloor \delta/2 \rfloor} \mathcal{T}'$.

   One can then show that the internal d-simplices in the d-supermersion set of a $(2d+1)$-dimensional triangulation $\mathcal{T}$ are given by the summands of the corresponding d-maximal green sequence G which are not projective or shifted projective.

   Hence, $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.

The "no-gap" conjecture

In [BDP14], Brüstle, Dupont, and Perotin conjectured that there was no gap in the set of lengths of maximal green sequences of a hereditary algebra over an algebraically closed field.

This conjecture was proved in some types by Garver and McConville [GM19] and for all tame types by Hermes and Igusa [HI19].

If the two orders on equivalence classes of d-maximal green sequences from the theorem are equal, then whenever $\Sigma(G) \supseteq \Sigma(G')$ we have a series of increasing elementary polygonal deformations from G to G' (up to equivalence).

Since an increasing elementary polygonal deformation changes the length of the d-maximal green sequence by 1, there are therefore no gaps in the lengths of maximal green sequences between G and G'.

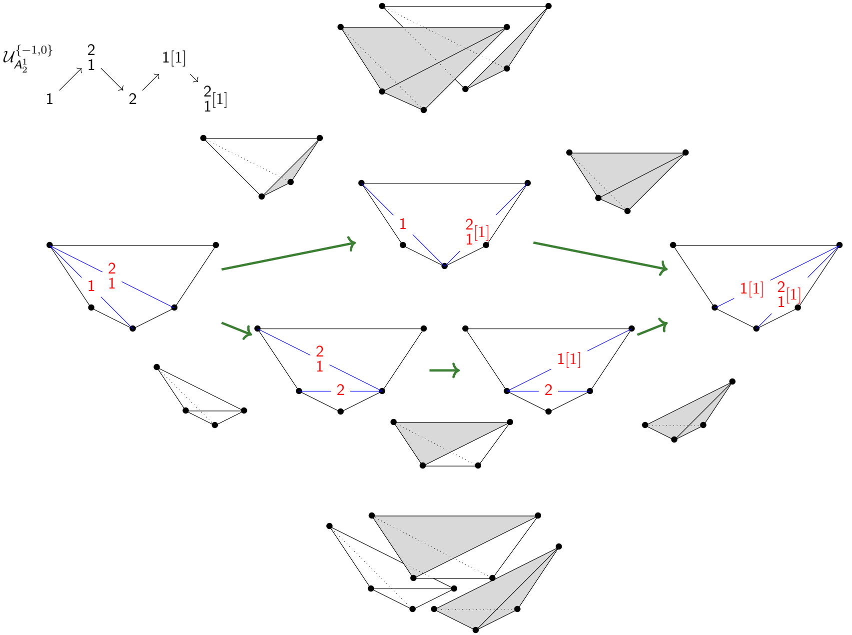
Consequences

Because we know from Edelman and Reiner that the higher Stasheff–Tamari orders are equal and are lattices for $\delta \leqslant 3$, we obtain the following result.

Corollary ([Wil21a])

The two orders on $\widetilde{\mathcal{MG}}_1(A_n)$ are equal and are lattices.

Illustration in odd dimensions



7. Equality of the higher Stasheff–Tamari orders

Introduction

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(m, \delta)$. In order to show that $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$, we need to show that if $\mathcal{T} <_2 \mathcal{T}'$, then there exists an increasing bistellar flip $\mathcal{T}''$ of $\mathcal{T}$ such that $\mathcal{T}'' \leqslant_2 \mathcal{T}'$.

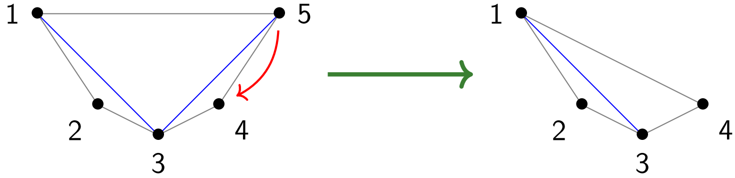
This gives us $\mathcal{T} \lessdot_1 \mathcal{T}'' \leqslant_2 \mathcal{T}'$. Then one can inductively construct a sequence of bistellar flips $\mathcal{T} = \mathcal{T}_0 \lessdot_1 \mathcal{T}'' = \mathcal{T}_1 \lessdot_1 \cdots \lessdot_1 \mathcal{T}_r = \mathcal{T}'$, giving $\mathcal{T} \leqslant_1 \mathcal{T}'$.

The problem is that bistellar flips are quite hard to find.

Our strategy is to use induction on the number of vertices of the cyclic polytope.

Contracting triangulations of cyclic polytopes

We consider the contraction operation $[m-1 \leftarrow m]$. Given a triangulation $\mathcal{T}$ of $C(m, \delta)$, $\mathcal{T}[m-1 \leftarrow m]$ is the triangulation of $C(m-1, \delta)$ which results from moving the vertex m along the moment curve until it coincides with the vertex $m-1$.



Main idea

We begin with two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $C(m, \delta)$ such that $\mathcal{T} <_2 \mathcal{T}'$.

We consider the contractions. We have $\mathcal{T}[m-1 \leftarrow m] \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$.

If $\mathcal{T}[m-1 \leftarrow m] = \mathcal{T}'[m-1 \leftarrow m]$, then we need to consider other contractions. Otherwise, the induction hypothesis tells us that there is a triangulation $\mathcal{U}$ of $C(m-1, \delta)$ such that $\mathcal{T}[m-1 \leftarrow m] \lessdot_1 \mathcal{U} \leqslant_2 \mathcal{T}'[m-1 \leftarrow m]$.

The increasing bistellar flip from $\mathcal{T}[m-1 \leftarrow m]$ to $\mathcal{U}$ happens inside some subpolytope congruent to $C(\delta+2, \delta)$.

When we expand back to $\mathcal{T}$, this subpolytope either remains congruent to $C(\delta+2, \delta)$, or expands to be congruent to $C(\delta+3, \delta)$.

In either case, we look inside this subpolytope to find an increasing bistellar flip $\mathcal{T}''$ of $\mathcal{T}$. It can be shown that $\mathcal{T}'' \leqslant_2 \mathcal{T}'$.

Result and algebraic consequences

Theorem ([Wil21b])

Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(m, \delta)$. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.

Corollary

The orders on d-silting complexes and equivalence classes of d-maximal green sequences discussed earlier are equal for A_n^d.

8. Mutation

Overview

Cluster categories were introduced for hereditary algebras in [Bua+06] in order to categorify cluster algebras. Clusters of the cluster algebra correspond to so-called cluster-tilting objects in the cluster category.

Higher cluster categories were introduced in [OT12] for d-representation-finite d-hereditary algebras.

For the classical case $d = 1$, cluster-tilting objects can be mutated at every summand, but this is not in general true for $d > 1$.

In this section, we look at a criterion for mutating summands of cluster-tilting objects in higher cluster categories.

Derived categories

Given a triangulated category $\mathcal{D}$, a functorially finite subcategory $\mathcal{C}$ of $\mathcal{D}$ is called *d-cluster-tilting* if

$$\begin{aligned}\mathcal{C} &= \{\, X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \mathrm{Hom}_{\mathcal{D}}(X, Y[i]) = 0 \,\} \\ &= \{\, X \in \mathcal{D} : \forall i \in [d-1], \forall Y \in \mathcal{C}, \mathrm{Hom}_{\mathcal{D}}(Y, X[i]) = 0 \,\}.\end{aligned}$$

Theorem ([Iya11, Theorem 1.23])

Let Λ be a d-representation-finite d-hereditary algebra with unique basic d-cluster-tilting module M. Then

$$\mathcal{U}_\Lambda := \mathrm{add}\{\, M[i] : i \in \mathbb{Z} \,\}$$

is a d-cluster-tilting subcategory of $D^b(\mathrm{mod}\,\Lambda)$.

Cluster categories

We denote by

$$\nu := D\Lambda \otimes_{\Lambda}^{\mathbf{L}} - \cong D\mathbf{R}\operatorname{Hom}_{\Lambda}\colon \mathcal{D}_{\Lambda} \to \mathcal{D}_{\Lambda},$$

the derived Nakayama functor.

Given a d-representation-finite d-hereditary algebra Λ, the *cluster category* of Λ is defined to be the orbit category [OT12, Definition 5.22]

$$\mathcal{O}_{\Lambda} = \frac{\mathcal{U}_{\Lambda}}{\nu[-2d]}\,.$$

For $d = 1$, this coincides with the classical cluster category of [Bua+06].

Cluster-tilting objects

Definition ([OT12, Definition 5.3])

An object $T \in \mathcal{O}_\Lambda$ is *cluster-tilting* if

1. $\mathrm{Hom}_{\mathcal{O}_\Lambda}(T, T[d]) = 0$, and
2. any $X \in \mathcal{O}_\Lambda$ occurs in a $(d+2)$-angle

$$X[-d] \to T_d \to T_{d-1} \to \cdots \to T_1 \to T_0 \to X$$

with $T_i \in \mathrm{add}\, T$.

Theorem ([OT12])

There are bijections between:

- *Triangulations of* $C(n+2d+1, 2d)$,
- *Basic cluster-tilting objects in* $\mathcal{O}_{A_n^d}$,
- *Non-intertwining subsets of* ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ *of size* $\binom{n+d-1}{d}$.

Higher cluster-tilted algebras

Theorem ([OT12, Theorem 5.6])

Let T be a cluster-tilting object in $\mathcal{O}_\Lambda$ and set $\Gamma := \mathrm{End}_{\mathcal{O}_\Lambda} T$. Then the functor

$$\mathrm{Hom}_{\mathcal{O}_\Lambda}(T, -) \colon \mathcal{O}_\Lambda \to \mathrm{mod}\,\Gamma$$

induces a fully faithful embedding

$$\mathcal{O}_\Lambda/(T[d]) \hookrightarrow \mathrm{mod}\,\Gamma,$$

where $(T[d])$ denotes the ideal of all morphisms factoring through $\mathrm{add}\, T[d]$. The image of this functor is a d-cluster-tilting subcategory $\mathcal{M}$ of $\mathrm{mod}\,\Gamma$.

Higher cluster-tilted algebras

Since $[d]$ is an automorphism of $\mathcal{O}_\Lambda$, we may restate this theorem as follows.

Theorem

Let T be a cluster-tilting object in $\mathcal{O}_\Lambda$ and set $\Gamma := \operatorname{End}_{\mathcal{O}_\Lambda} T$. Then the functor

$$\operatorname{Hom}_{\mathcal{O}_\Lambda}(T, -[d]) \colon \mathcal{O}_\Lambda \to \operatorname{mod} \Gamma$$

induces a fully faithful embedding

$$\mathcal{O}_\Lambda/(T) \hookrightarrow \operatorname{mod} \Gamma.$$

The image of this functor is a d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod} \Gamma$. In particular, Γ is weakly d-representation-finite.

Mutating cluster-tilting objects

Given a cluster-tilting object $T = E \oplus X$ in $\mathcal{O}_\Lambda$, where X is indecomposable, we say that T is *mutable* at X if there is a cluster-tilting object $E \oplus Y$ with $X \ncong Y$.

In order for T to be mutable at X, it is necessary and sufficient that there is a Y such that $\mathrm{Hom}_{\mathcal{O}_\Lambda}(E, Y[d]) = 0$ and $\mathrm{Hom}_{\mathcal{O}_\Lambda}(X, Y[d]) \neq 0$ [OT12].

In this case, $E \oplus Y$ is also a cluster-tilting object [OT12].

Criterion for mutation

Theorem

Let T be a basic cluster-tilting object in $\mathcal{O}_{A_n^d}$ with indecomposable summand X. Then T is mutable at X if and only if the d-cluster-tilting subcategory $\mathcal{M}$ of $\operatorname{mod}\Gamma$ contains the simple Γ-module corresponding to X.

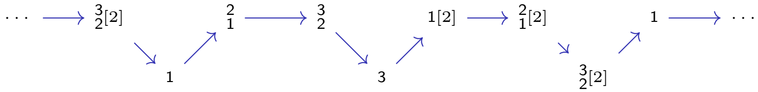
$T = E \oplus X$ is mutable at X if and only if there exists Y such that $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(E, Y[d]) = 0$ and $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(X, Y[d]) \neq 0$.

We have that $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(X, Y[d])$ is therefore a one-dimensional vector space. [OT12]

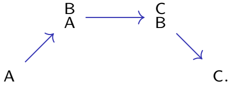
Hence, $\operatorname{Hom}_{\mathcal{O}_{A_n^d}}(T, Y[d])$ is the simple Γ-module corresponding to X.

Criterion for mutation: example

Take the cluster-tilting object $T = 1 \oplus \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$ in $\mathcal{O}_{A_2^2}$



If we label the summands A, B, and C respectively, then the 2-cluster-tilting subcategory of $\operatorname{mod} \operatorname{End} T$ is



This shows that the mutable summands are A and C.

どうもありがとうございました!

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