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Based modules over the ıquantum groups of type AI

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Plan

1. Background ((ı-)quantum groups, cells)
2. ıquantum group of type AI
3. 1st main thm. (ıcanonical bases)
4. 2nd main thm. (branching rule for $\mathfrak{so}_n \subset \mathfrak{sl}_n$)

1. Background

$\mathfrak{g}$: fin. dim'l semisimple Lie alg. / $\mathbb{C}$

(e.g. $\mathfrak{g} = \mathfrak{sl}_n = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid \mathrm{tr}\, X = 0\}$)

f.d. $\mathfrak{g}$-mod's are completely reducible

$P^+ := \{$dominant integral weights$\}$

(e.g. $P^+ = \{\lambda = (\lambda_1, \cdots, \lambda_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1} \mid \lambda_i \geq \lambda_{i+1}\}$)

$\{$f.d. irreducible $\mathfrak{g}$-mod's$\}/_{\text{isom.}} \overset{1:1}{\longleftrightarrow} P^+$

$\ni$ $V(\lambda)$: irr. $\mathfrak{g}$-mod. of highest weight λ $\longleftarrow$ λ $\in$

$\mathfrak{g}$-mod. structure of $V(\lambda)$?

$U_q(\mathfrak{g})$: quantum group / $\mathbb{C}(q)$

$=$ q-deformation of $U(\mathfrak{g})$

($\lim_{q \to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$: classical limit)

$U(\mathfrak{g})$: universal enveloping alg.

$=$ associative alg. approximation of $\mathfrak{g}$

$\mathfrak{g}$-mod. $=$ $U(\mathfrak{g})$-mod.

$$
\begin{tikzcd}
U_q(\mathfrak{g})\text{-mod} & \\
\mathcal{C} \arrow[u, phantom, "\cup" description] \arrow[r, "\sim", "\text{classical limit}"'] & \mathfrak{g}\text{-mod}_{f.d.} \\
V_q(\lambda) \arrow[u, phantom, "\cup" description] & V(\lambda) \arrow[u, phantom, "\cup" description] \arrow[l, mapsto]
\end{tikzcd}
$$

$\uparrow$

irr. $U_q(\mathfrak{g})$-mod. of h.w. λ

$U_q(\mathfrak{g})$-str. of $V_q(\lambda)$?

$V_q(\lambda)$ has a distinguished basis

Lusztig's canonical basis (CB)

= Kashiwara's global crystal basis

$\mathbb{A} := \mathbb{Z}[q, q^{-1}] \subset \mathbb{C}(q)$

$^\exists U_q(\mathfrak{g})_{\mathbb{A}} \subset U_q(\mathfrak{g})$: free $\mathbb{A}$-mod., $\mathbb{A}$-subalg,

$U_q(\mathfrak{g})_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{C}(q) = U_q(\mathfrak{g})$

$V_q(\lambda)_{\mathbb{A}} :=$ $\mathbb{A}$-span of the CB.

- $V_q(\lambda)_{\mathbb{A}}$ is a $U_q(\mathfrak{g})_{\mathbb{A}}$-submod.

  free $\mathbb{A}$-mod.

  $V_q(\lambda)_{\mathbb{A}} \otimes_{\mathbb{A}} \mathbb{C}(q) = V_q(\lambda)$

CB $\xrightarrow{q \to \infty}$ crystal basis

↑

combinatorial feature

$\theta : \mathfrak{g} \to \mathfrak{g}$; Lie alg. automorphism s.t. $\theta^2 = \mathrm{id}_{\mathfrak{g}}$

(e.g. $E_{i,j} \mapsto E_{j,i}$ $(i \neq j)$ $E_{i,i} - E_{i+1,i+1} \mapsto -E_{i,i} + E_{i+1,i+1}$)

$\mathfrak{k} := \mathfrak{g}^{\theta} = \{ X \in \mathfrak{g} \mid \theta(X) = X \}$

(e.g. $\mathfrak{k} \simeq \mathfrak{so}_n$)

$(\mathfrak{g}, \mathfrak{k})$ is called a symmetric pair

$U^{\iota}(\mathfrak{k})$: ιquantum group

$= \begin{cases} \cdot\ q\text{-deformation of } U(\mathfrak{k}) \\ \cdot\ \text{right coideal of } U_q(\mathfrak{g}) \\ \cdot\ \text{max'l among those satisfying} \end{cases}$

$(U_q(\mathfrak{g}), U^{\iota}(\mathfrak{k}))$ is called quantum symm. pair

Letzter : comprehensive construction of $U^{\iota}(\mathfrak{k})$

Earlier examples by Noumi and others

ιquantum groups are generalizations of quantum grp's:

$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$: semisimple

$I = I_1 \amalg I_2$: Dynkin diag. $I_1 = I_2$

$e_i^{(j)}, f_i^{(j)}, h_i^{(j)}$: Chevalley generators for $\mathfrak{g}_j$ $(j = 1, 2)$

$i \in I_1$

$\theta : \mathfrak{g} \to \mathfrak{g}$; $e_i^{(1)} \mapsto f_i^{(2)}$, $f_i^{(1)} \mapsto e_i^{(2)}$, $h_i^{(1)} \mapsto -h_i^{(2)}$

$$\leadsto \mathfrak{k} \simeq \mathfrak{g}_1 \ ; \quad \begin{aligned} f_i^{(1)} + e_i^{(2)} &\longleftrightarrow f_i^{(1)} \\ f_i^{(2)} + e_i^{(1)} &\longleftrightarrow e_i^{(1)} \\ h_i^{(1)} - h_i^{(2)} &\longleftrightarrow h_i^{(1)} \end{aligned}$$

In this case,
$$\begin{array}{ccc} U_q(\mathfrak{g}) & \simeq & U_q(\mathfrak{g}_1) \otimes U_q(\mathfrak{g}_1) \\ \cup & & \cup \\ U^\iota(\mathfrak{k}) & \simeq & \Delta(U_q(\mathfrak{g}_1)) \simeq U_q(\mathfrak{g}_1) \end{array}$$

Slogan (ıprogram by Bao-Wang)

Generalize what are known about quantum groups to ıquantum groups

Achievements this far

- K-matrix (ıver. of R-matrix)
- ıcanonical basis (ıver. of CB)
- q-Schur duality
- Kazhdan-Lusztig theory
- geom. construction
- Hall alg. construction
- braid group action
- ⋮

Today

Improve ıCB theory

"cellular basis"

A : alg, M : A-mod. B : basis of M

For $b, b' \in B$, $b' \leq b \;:\Leftrightarrow\; b \in Ab'$

$b' \sim b \;:\Leftrightarrow\; b' \leq b$ and $b \leq b'$

An equivalent class is called a cell

For $C \in B/\sim$, set

$M[\underset{>}{\geq} C] := \mathrm{Span}\{ b' \mid b' \underset{>}{\geq} b \;\; {}^{\forall} b \in C \}$

$\rightarrow M(C) := M[\geq C] / M[> C]$ is an A-mod : cell mod.

Ex.

$A = U_q(\mathfrak{g})$, M : $\underset{\underset{U_q(\mathfrak{g})\text{-mod. w/. CB}}{\shortparallel}}{\text{based}}$ mod. $B = CB$ of M

$\rightarrow$ each cell modules are irreducible

$$\begin{array}{ccc} M(C) & \simeq & V_q(\lambda) \\ \cup & & \cup \\ \{ b + M[> C] \mid b \in C \} & \longleftrightarrow & CB \end{array}$$

Problems

- "$\imath$canonical bases" are known to exist only for f.d. $U_q(\mathfrak{g})$-modules

  Can we construct the $\imath CB$ for irr. $U^\imath(\mathfrak{k})$-modules?

- A cell module of the $\imath CB$ for a $U_q(\mathfrak{g})$-mod. is NOT irr. $U^\imath(\mathfrak{k})$-mod. in general

  Can we modify $\imath CB$?

2. ıquantum group of type AI

$(\mathfrak{g} = \mathfrak{sl}_n,\ \mathfrak{k} = \mathfrak{so}_n)$: symm. pair of type AI

$U_q(\mathfrak{g}) = \langle E_i, F_i, K_i^{\pm 1} \mid i = 1, \dots, n-1 \rangle / \sim$

$U^\imath(\mathfrak{k})$ = subalg. of $U_q(\mathfrak{g})$ generated by $F_i + q^{-1} E_i K_i^{-1}$

$(i = 1, \dots, n-1)$

$$\simeq \langle B_1, \dots, B_{n-1} \rangle \Big/ \begin{pmatrix} B_i B_j = B_j B_i & \text{if } |i-j| > 1 \\ B_i^2 B_j - (q+q^{-1}) B_i B_j B_i + B_j B_i^2 = B_j & \text{if } |i-j| = 1 \end{pmatrix}$$

$F_i + q^{-1} E_i K_i^{-1} \longleftrightarrow B_i$

$$m := \operatorname{rank} \mathfrak{k} = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

$U^\imath(\mathfrak{k})^0 := \langle B_1, B_3, \dots, B_{2m-1} \rangle$ is a comm. subalg. of $U^\imath(\mathfrak{k})$

Def.
M : f.d. $U^\imath(\mathfrak{k})$-mod. is a classical wt. mod.
:⟺ $U^\imath(\mathfrak{k})^0$ acts on M semisimply
s.t. eigenvalues of B_{2j-1} are $\frac{q^a - q^{-a}}{q - q^{-1}}$ $(a \in \frac{1}{2}\mathbb{Z})$

$$M = \bigoplus_{\mu} M_\mu$$

$\mu = (\mu_1, \cdots, \mu_m)$ $\quad M_\mu = \{ m \in M \mid B_{2j-1} m = [\mu_j] m \ \ {}^\forall j = 1, \cdots, m \}$

Thm [W. 19]

- f.d. classical wt. $U'(k)$-mod. is completely reducible
- irr. f.d. cl. wt. $U'(k)$-mod. is classified by P_k^+

$$P_k^+ = \begin{cases} \left\{ \nu = (\nu_1, \nu_3, \cdots, \nu_{2m-1}) \in (\tfrac{1}{2}\mathbb{Z})^m \;\middle|\; \begin{array}{l} \nu_{2i-1} - \nu_{2i+1} \in \mathbb{Z}_{\geq 0} \\ \nu_{2m-1} \geq 0 \end{array} \right\} & \text{if } n: \text{odd} \\ \left\{ \nu = (\nu_1, \nu_3, \cdots, \nu_{2m-1}) \in (\tfrac{1}{2}\mathbb{Z})^m \;\middle|\; \begin{array}{l} \nu_{2i-1} - \nu_{2i+1} \in \mathbb{Z}_{\geq 0} \\ \nu_{2m-3} - |\nu_{2m-1}| \in \mathbb{Z}_{\geq 0} \end{array} \right\} & \text{if } n: \text{even} \end{cases}$$

$\nu \in P_k^+$

$\updownarrow$

$$V^c(\nu) = \bigoplus_{\xi \in P_k} V^c(\nu)_\xi \qquad \dim V^c(\nu)_\nu = 1$$

dominance

$$\dim V^c(\nu)_\xi = 0 \text{ unless } \xi \leq \nu$$

- $\lim_{q \to 1} V^c(\nu) = V(\nu)$: f.d. irr. k-mod. of h.wt. ν

3. 1st main thm.

Thm [W.]

(1) Let $\nu \in P_k^+ \cap \mathbb{Z}^m$.

Then, $V^\imath(\nu)$ has an $\imath$CB $B^\imath(\nu)$.

(2) Let M be a based $U_q(\mathfrak{g})$-mod. w/. a CB B,

$B^\imath$ the associated $\imath$CB

Then, $\exists B' \subset \mathbb{C}B^\imath$ s.t.

- (M, B') is a based $U^\imath(k)$-mod
- $\forall$cell C $\exists ! \nu \in P_k^+ \cap \mathbb{Z}^m$ s.t. $(M(C), C + M[> C]) \simeq (V^\imath(\nu), B^\imath(\nu))$

4. 2nd main thm.

Let $\lambda \in P^+$ and consider $V_q(\lambda)$: irr. $U_q(\mathfrak{g})$-mod.

Fact : $V_q(\lambda)$ is a classical wt. $U^\iota(\mathfrak{k})$-mod.

$$\rightarrow V_q(\lambda) \simeq \bigoplus_{\nu \in P_{\mathfrak{k}}^+} V^\iota(\nu)^{\oplus m_{\lambda,\nu}} \quad \text{as } U^\iota(\mathfrak{k})\text{-mod's}$$

$$\left(m_{\lambda,\nu} = \dim_{U^\iota(\mathfrak{k})}(V^\iota(\nu), V_q(\lambda)) = \dim_{\mathfrak{k}}(\underbrace{V(\nu)}_{\text{irr. } \mathfrak{k}\text{-mod.}}, \underbrace{V(\lambda)}_{\text{irr. } \mathfrak{g}\text{-mod.}}) \right)$$

By 1st main thm, $V_q(\lambda)$ has an ιCB $B^\iota(\lambda)$

$$B^\iota(\lambda) \xrightarrow{q \to \infty} \mathcal{B}^\iota(\lambda)$$

Fact : $\mathbb{C}\mathcal{B}^\iota(\lambda) = \mathbb{C}\underbrace{\mathcal{B}(\lambda)}_{\text{crystal basis of } V_q(\lambda)}$

$$\mathbb{C}\mathcal{B}^\iota(\lambda) = \bigoplus_{\nu \in P_{\mathfrak{k}}^+} \mathbb{C}\underbrace{\mathcal{B}^\iota(\nu)}_{\lim_{q \to \infty} B^\iota(\nu)}{}^{\oplus m_{\lambda,\nu}}$$

- $m_{\lambda,\nu}$ = dim. of the subspace of $\mathbb{C}B^L(\lambda)$ spanned by the h.w.v. of h.w. ν

as an ıcrystal basis element

Thm 2

For $\lambda \in P^+$, $\nu \in P_k^+ \cap \mathbb{Z}^m$, we have

$$m_{\lambda,\nu} = \#\left\{ b \in B(\lambda) \;\middle|\; \begin{array}{ll} \widetilde{B}_{2i} b = 0 & \forall i = 1, \dots, m \\ \widetilde{B}_{2i+1} (\widetilde{B}_{2i} \widetilde{B}_{2i-1})^{|\nu_{2i+1}|} b = 0 & \forall i = 1, \dots, m-1 \\ \widetilde{B}_{2i+1} (\widetilde{B}_{2i} \widetilde{B}_{2i-1})^{n} b \neq 0 & \forall n < |\nu_{2i+1}| \end{array} \right\}$$

where

$$\widetilde{B}_i b := \begin{cases} \widetilde{E}_i b & \text{if } \varphi_i(b) \text{ is even} \\ \widetilde{F}_i b & \text{if } \varphi_i(b) \text{ is odd} \end{cases}$$

$(b \in B(\lambda))$