Flat cotorsion modules over Noether algebras

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<u>Aim</u> Classify all flat cotorsion modules for Noether algebras in terms of prime ideals.

(Generalization of [Enochs 1984] for comm noeth rings)

§1 Flat cotorsion modules

A : ring. $\operatorname{Mod} A := \{\text{right modules}\}$.

<u>Def</u> $M \in \operatorname{Mod} A$: flat

$:\Longleftrightarrow M \underset{A}{\otimes} - : \operatorname{Mod} A^{op} \to \operatorname{Mod} \mathbb{Z}$: exact.

$\operatorname{Flat} A := \{\text{flat in } \operatorname{Mod} A\}$

$M \in \operatorname{Mod} A$: cotorsion

$:\Longleftrightarrow \operatorname{Ext}^1_A(\operatorname{Flat} A, M) = 0$.

$\operatorname{Cot} A := \{\text{cotorsion in } \operatorname{Mod} A\}$.

$\operatorname{FlCot} A := \operatorname{Flat} A \cap \operatorname{Cot} A$.

Def $\mathcal{A}$: abelian cat, $\mathcal{X}, \mathcal{Y} \underset{\text{(full)}}{\subset} \mathcal{A}$.

$(\mathcal{X}, \mathcal{Y})$: cotorsion pair

$$:\Longleftrightarrow \begin{cases} \mathcal{X} = \{ M \mid \mathrm{Ext}^1(M, \mathcal{Y}) = 0 \} \\ \mathcal{Y} = \{ M \mid \mathrm{Ext}^1(\mathcal{X}, M) = 0 \}. \end{cases}$$

Moreover

hereditary $:\Longleftrightarrow \mathrm{Ext}^{>0}(\mathcal{X}, \mathcal{Y}) = 0$.

Complete $:\Longleftrightarrow \forall M \in \mathcal{A}$,

$$\begin{array}{l} \exists\, 0 \to Y \to X \to M \to 0 \\ \exists\, 0 \to M \to Y' \to X' \to 0 \end{array} \quad \begin{pmatrix} X, X' \in \mathcal{X} \\ Y, Y' \in \mathcal{Y} \end{pmatrix}$$

Fact (Flat A, Cot A) is a complete hereditary cotorsion pair.

$\Uparrow$

Thm (Flat cover conjecture) $\forall A$: ring, $\forall M \in \mathrm{Mod}\,A$, $\exists F_A(M) \to M$: flat cover. (right min'l (Flat A)-approximation)

$\exists M \to C_A(M)$: cotorsion envelope.

Solved by [Bican–El Bashir–Enochs 2001].

Remark Flat cover conj holds for QCohX. [Enochs–Estrada 2005]

Why flat cotorsion?

$$
\begin{tikzcd}
K^-(\mathrm{Proj}\,A) \arrow[r, "\sim"] \arrow[d, phantom, "\cap" rotate=90] & D^-(\mathrm{Mod}\,A) \arrow[d, phantom, "\cap" rotate=90] \\
K_{\text{K-proj}}(\mathrm{Proj}\,A) \arrow[r, "\sim"] \arrow[d, phantom, "\cap" rotate=90] & D(\mathrm{Mod}\,A) \arrow[d, phantom, "\wr"] \\
K(\mathrm{Proj}\,A) & \left(K_{\text{K-proj}\,A}(\mathrm{Mod}\,A)\right)
\end{tikzcd}
$$

[Spaltenstein 1988].

<u>Thm</u> $K_{\text{K-flat}}(\mathrm{FlCot}\,A) \xrightarrow{\sim} D(\mathrm{Mod}\,A)$

([Gillespie 2004] + [Bazzoni–(Cortés-Izurdiaga)–Estrada 2020]

$\exists$ flat model str on $C(\mathrm{Mod}\,A)$

+ [Nakamura–Thompson 2020].)

§2 Noether algebras

R: comm noeth ring.

A: Noether R-alg.

i.e. $R \subset Z(A) \subset A$ & A is fin gen as an R-module.

subring ↑ center

Then A is left noeth & right noeth.

$$\{\text{Noeth algs}\}$$

$$\{\text{comm noeth ring}\} \subset \{\text{Noeth algs}\} \supset \{\text{fin.dim algs/fields}\}$$

$A = R$ $\qquad\qquad$ $R = k$: field.

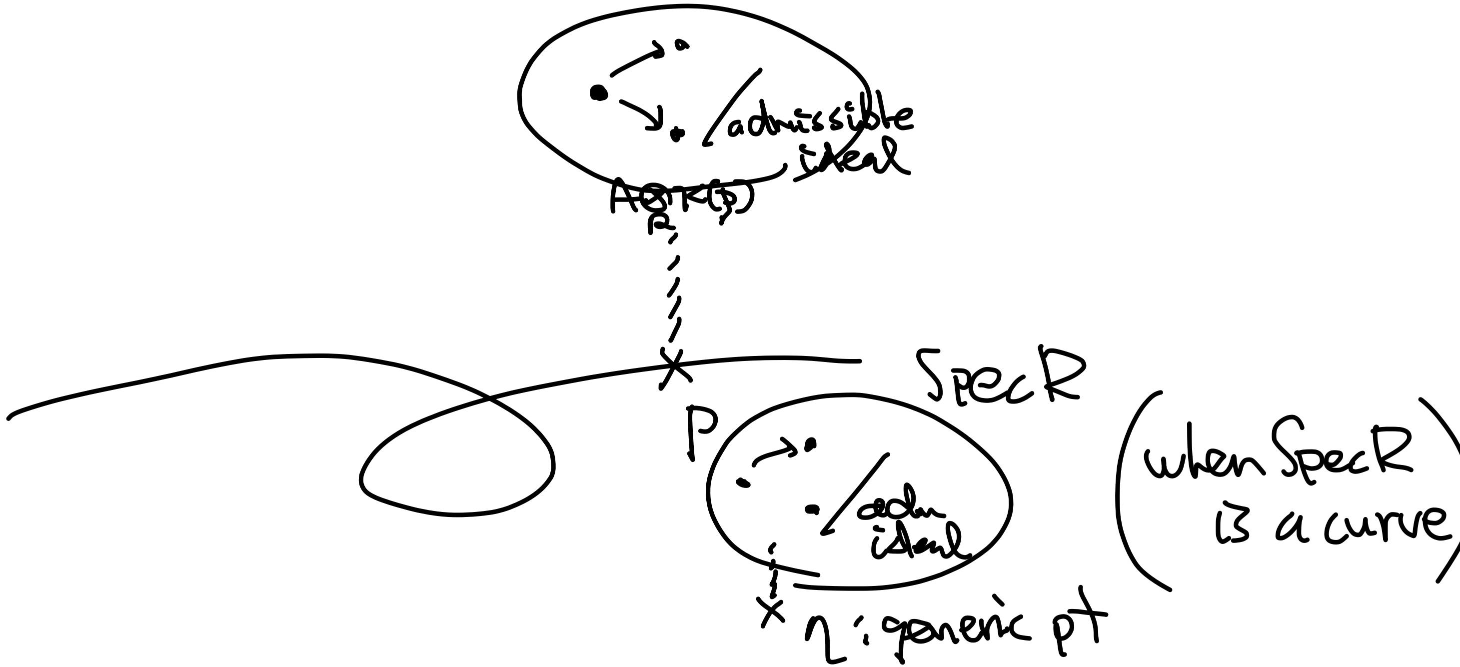
Def $P \subsetneq A$: ideal (two-sided)

P : prime ideal $:\Longleftrightarrow$ $[\forall a, b \in A,\ aAb \subset P \Rightarrow a \in P$ or $b \in P]$.

$$\begin{array}{ccc} A & & \operatorname{Spec} A \\ \uparrow & & \downarrow \\ R & & \operatorname{Spec} R \end{array} \qquad \operatorname{Spec} A \overset{\text{as a set}}{=} \coprod_{p \in \operatorname{Spec} R} \underbrace{\{P \mid P \cap R = p\}}$$

$$\begin{array}{c} \| \text{ identify} \\ \operatorname{Spec} A \otimes_R \underbrace{\kappa(p)} \overset{1-1}{\longleftrightarrow} \{\text{simple in } \operatorname{Mod} A \otimes_R \kappa(p)\}/\cong \\ \wr\wr \\ \dfrac{R_p}{pR_p} \end{array}$$

fin. dim $\kappa(p)$-alg.



<u>Fact</u> $\mathrm{Spec}\, A \overset{1-1}{\longleftrightarrow} \{\text{indec injective in } \mathrm{Mod}\, A\}/\cong$

$$\begin{array}{ccc} \mathrm{Spec}\, A & & \{\text{indec injective in } \mathrm{Mod}\, A\}/\cong \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathfrak{p} & \longmapsto & I_A(\mathfrak{p}) \end{array}$$

$\mathrm{Ass}\, I_A(\mathfrak{p}) = \{\mathfrak{p}\}$.

$E_A\left(\frac{A}{\mathfrak{p}}\right) \cong I_A(\mathfrak{p})^{n_\mathfrak{p}}$

$(n_\mathfrak{p} : \text{finite})$

§4 Flcot over Noeth algs

Where are flcot? $\forall M \in \mathrm{Mod} R$, $M \overset{\exists}{\hookrightarrow} \prod E$.

<u>Prop</u> Let E: injective cogenerator in $\mathrm{Mod} R$.

(e.g. R: local $E = E_R(R/m)$)

$$(-)^* := \mathrm{Hom}_R(-, E) : (\mathrm{Mod} A)^{op} \longrightarrow \mathrm{Mod}(A^{op}).$$

1. $\forall M \in \mathrm{Mod} A$, M^* is cotorsion.
2. M: flat $\iff$ M^*: inj.
3. M: inj $\iff$ M^*: flat. (& cotorsion)

Let $M \in \mathrm{FlCot}\, A$. Describe M!

Fix $E \in \mathrm{Mod}\, R$: inj cogen.

$M \longrightarrow M^{**}$ is a pure monomorphism, hence splits.

flcot $\Rightarrow$ pure-injective

$$M \lhd^{\oplus} M^{**} = \mathrm{Hom}_R(\underbrace{M^*}_{\text{inj in } \mathrm{Mod} A^{op}}, E)$$

$$M^* \cong \bigoplus_{P \in \mathrm{Spec} A} I_{A^{op}}(P)^{\oplus C_P}.$$

$$= \prod_P \mathrm{Hom}_R(\underbrace{I_{A^{op}}(P)}_{\cong\, I_{A^{op}}(P) \otimes_R R_p}, E)^{C_P}$$

$p := P \cap R.$

$$= \prod_{\mathfrak{p}} \mathrm{Hom}_R\left(I_{A^{op}}(P) \otimes_R R_{\mathfrak{p}}, E^{C_{\mathfrak{p}}}\right)$$

$$= \prod_{\mathfrak{p}} \mathrm{Hom}_R\Big(I_{A^{op}}(P), \underbrace{\mathrm{Hom}_R(R_{\mathfrak{p}}, E^{C_{\mathfrak{p}}})}_{\text{inj in } \mathrm{Mod}\, R_{\mathfrak{p}}}\Big)$$

$$\cong \bigoplus_{\mathfrak{q} \subset \mathfrak{p}} E_R\left(\frac{R}{\mathfrak{q}}\right)^{\oplus}$$

$$\Big(\mathrm{Hom}_R\big(\underbrace{I_{A^{op}}(P)}_{\mathfrak{p}\text{-torsion}}, E_R\big(\tfrac{R}{\mathfrak{q}}\big)\big) = 0 \text{ unless } \mathfrak{p} \subset \mathfrak{q}. \Big)$$

$$= \prod_{P \in \mathrm{Spec}\, A} \mathrm{Hom}_R\left(I_{A^{op}}(P), E_R\left(\frac{R}{\mathfrak{p}}\right)^{\oplus}\right)$$

Thm $M \in \operatorname{Mod} A$: flat cotorsion

$$\iff M \cong \prod_{p \in \operatorname{Spec} A} \operatorname{Hom}_R\left(I_{A^{op}}(p), E_R\left(\frac{R}{p \cap R}\right)^{\oplus B_p}\right).$$

$$\cong \left(\underbrace{\operatorname{Hom}_R\left(I_{A^{op}}(p), E_R\left(\frac{R}{p\cap R}\right)\right)}_{!! \; T_A(p) \,\oplus\, (A_p)^\wedge_p}\right)^{\oplus B_p}{}^\wedge_p$$

$$\operatorname{Hom}_R\left(\operatorname{Hom}_R\left(A, E_R\left(\frac{R}{p}\right)\right), E_R\left(\frac{R}{p}\right)\right) = (A_p)^\wedge_p$$

The cardinality of B_p is determined by M.

$(-)^\wedge_p$: p-adic completion
$\operatorname{Mod} A \longrightarrow \operatorname{Mod} A$.

[Enochs 1984] for $R = A$, [Kanda–Nakamura] in general.

Difficult part: M is not just a direct summand, but <u>is</u> of that form.

<u>Cor</u> $\{\text{indec flat in } \operatorname{Mod} A\}/\cong \xleftrightarrow{1-1} \operatorname{Spec} A$.

$$\begin{array}{ccc} T_A(P) & \longleftarrow\!\!\shortmid & P \end{array}$$

Compute $T_A(P)$?

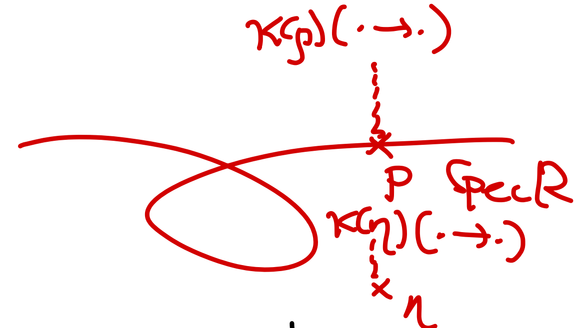
<u>Prp</u> [Kanda–Nakamura]

$\forall \mathfrak{p} \in \operatorname{Spec} R$,

$$(A_{\mathfrak{p}})^{\wedge}_{\mathfrak{p}} \cong \bigoplus_{P \cap R = \mathfrak{p}} T_A(P)^{n_P} \text{ in } \operatorname{Mod} A.$$

$\underline{\text{Ex}}$ $A=\begin{pmatrix}R & 0\\ R & R\end{pmatrix}=R(\cdot\to\cdot)$

$$(A_p)^\wedge_p=\begin{pmatrix}\hat{R}_p & \\ \hat{R}_p & \hat{R}_p\end{pmatrix}=\begin{matrix}(\hat{R}_p\ \ 0)\\ \oplus\\ (\hat{R}_p\ \ \hat{R}_p)\end{matrix},$$



$$\{\text{indec flat in Mod}\,A\}/\cong\ =\{(\hat{R}_p\ \ 0),\ (\hat{R}_p\ \ \hat{R}_p)\mid p\in\operatorname{Spec}R\}.$$

§5 Ziegler spectrum

$\underline{\text{Def}}$ $0\to L\xrightarrow{\text{pure monomorphism}}M\to N\to 0$ in $\text{Mod}\,A$ is pure-exact

$$:\Longleftrightarrow 0\to L\underset{A}{\otimes}U\to M\underset{A}{\otimes}U\to N\underset{A}{\otimes}U\to 0 \ :\ \text{exact},$$

$$\forall U\in \text{Mod}\,A^{op}.$$

$I \in \operatorname{Mod} A$: pure-injective

$:\Longleftrightarrow$ All pure exact $0 \to I \to M \to N \to 0$ splits,

<u>Def</u> $Zg_A := \{\text{indec pure-injective in } \operatorname{Mod} A\}/\cong$.

: Ziegler spectrum.

$\exists$ topology with

fin. presented

open basis : $\{ (F) \mid F \in \operatorname{fp}(\operatorname{mod} A, \operatorname{Mod} \mathbb{Z}) \}$.

fin presented functor

$(F) = \{ I \in Zg_A \mid F(I) \neq 0 \}$

extension to Mod A
that presences $\varinjlim$ (filtered colim)

Thm [Herzog 1993] (Elementary duality)

$$\{\text{open subsets of } {}_2g_A\} \overset{1-1}{\longleftrightarrow} \{\text{open subsets of } {}_2g_{A^{op}}\}.$$

$$\begin{array}{ccc} \cup & & \cup \\ (F) & \longmapsto & (dF) \end{array}$$

Auslander–Gruson–Jensen duality.

(**Open Problem** ${}_2g_A \overset{\text{homeo}}{\cong} {}_2g_{A^{op}}$.)

$$\text{closed} \subset {}_2g_A \supset \text{closed}.$$

$$\text{flcot}_A \qquad\qquad \text{inj}_A$$

$$\| \qquad\qquad\qquad \|$$

$$\{\text{indec flcot}\}/\cong \qquad \{\text{indec inj}\}/\cong$$

closed $\subset {}_2g_A$.

$\updownarrow$ 1–1

definable ... $\prod$, $\varinjlim$, pure-sub

$\cap$

Mod A.

Thm [Herzog 1993]

Elementary duality induces $\mathrm{flcot}_A \overset{\text{homeo}}{\cong} \mathrm{inj}_{A^{op}} \overset{\text{homeo}}{\cong} \mathrm{Spec}\,A$.

(but the correspondence was not clear.)

Our result makes it explicit: $T_A(P) \longleftrightarrow I_{A^{op}}(P)$.

$$I_{A^{op}}(P) = \mathrm{Hom}_{\hat{R}_p}(-, E_R(R/p))$$

($p := P \cap R$ depends on P)

$\Phi \subset \mathrm{Spec}\,A$: open $\iff$ Φ : specialization-closed.

(Serre $\subset \mathrm{fp}(\mathrm{mod}\,A, \mathrm{Mod}\,\mathbb{Z}) \xleftrightarrow{1-1}$ open $\subset \mathrm{Zg}_A$.)