1. Intro
2. Gentle alg & HKK's surface model
3. CY completion & graded gentle algs
4. Correspondence between arcs & spherical objects

(joint work with Qiu-Zhou)

arXiv: 2006.00009

1. Intro

background $\boxed{\text{Homological mirror symmetry}}$

A模型の圏 ⥂ B模型の圏

	(幾何)	(代数)
"空間"	X : symplectic mfd	X^+ : cpx mfd A_X : non-commutative alg
対象	$X \supset L$ Lagrangian submfd	$E \to X^+$: ベクトル束 (連接層の複体) M : A_X-module
射	$L_1 \cap L_2$: intersection	$\mathrm{Ext}^i(M_1, M_2)$

おまけ

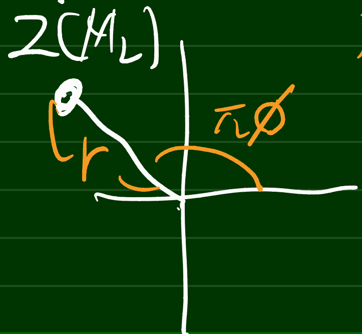
X : CY

Ω : X上のhol volume form	加群の安定性条件 Z
$\displaystyle\int_L \Omega$	$= Z(M_L) \in \mathbb{C}^*$

周期

Lの体積と角度

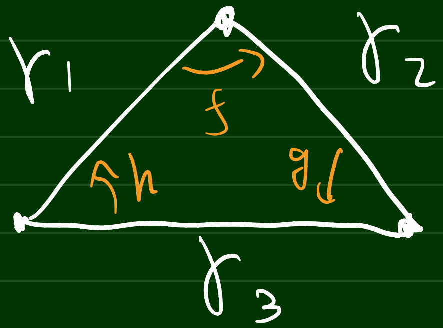
M_L : HMSでLに対応する加群



を念頭において.

A：曲面の幾何 $\longleftrightarrow$ B：gentle algebraや その CY-completion

曲線, arc $\longleftrightarrow$ module.



$\longleftrightarrow$

$$\begin{tikzcd}
M_{\gamma_1} \arrow[rr, "f"] & & M_{\gamma_2} \arrow[dl, "g"] \\
& M_{\gamma_3} \arrow[ul, rightsquigarrow, "h"] &
\end{tikzcd}$$

HKK (Haiden-Katzarkov-Kontsevich)

model of topological Fukaya

graded marked bordered surface $(\mathbb{S}, \mathbb{M}, \lambda)$ $\rightsquigarrow$ A_T : graded gentle alg.

$+$

$\mathbb{S}$ の full formal arc system T

Thm (HKK)

$\mathbb{S}$ のあるクラスの (graded) arc with local system $\overset{1:1}{\longleftrightarrow}$ $\mathrm{Per}\, A_T$ の indecomposable obj

{ HKKの結果の一部のリスト

Qiu

$\mathbb{S}'$ の三角形分割 T' $\rightsquigarrow$ $\Gamma_{T'}$: Ginzburg の CY-3 alg.

にも見える。

Thm (Qiu)

$\mathbb{S}'$ の closed arc (graded) $\overset{1:1}{\longleftrightarrow}$ $\mathcal{D}_{fd}(\Gamma_{T'})$ の reachable spherical obj.

Theorem (I-Qiu-Zhou)

$\mathbb{S}, T \rightsquigarrow {}^{\exists}$CY-$\mathbb{X}$ alg $\Gamma^{\mathbb{X}}_{T}$, $\leftarrow$ CY-$\mathbb{X}$ completion of graded gentle

deformed

(log) $\mathbb{S}_\Delta$ の $\mathbb{Z}^2$-graded arc $\xleftrightarrow{1:1}$ $\mathcal{D}_{fd}(\Gamma^{\times}_{T})$ の reachable spherical obj

decoration

alg.

2. Gentle alg & HKK's surface model

デ ー タ

$(\mathbb{S}, \mathbb{M}, \lambda)$: graded marked surface

- $\mathbb{S}$: cpt, connected, oriented real 2-mfd with boundary.
- $\mathbb{M} \subset \partial \mathbb{S}$: finite pts

$\partial_1 \sqcup \partial_2 \sqcup \cdots \sqcup \partial_b \qquad \mathbb{M} \cap \partial_i \neq \emptyset$

boundary component

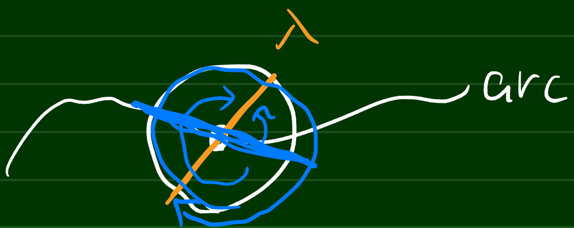
(各 boundary comp は少なくとも1つ $\mathbb{M}$ の点を含む)

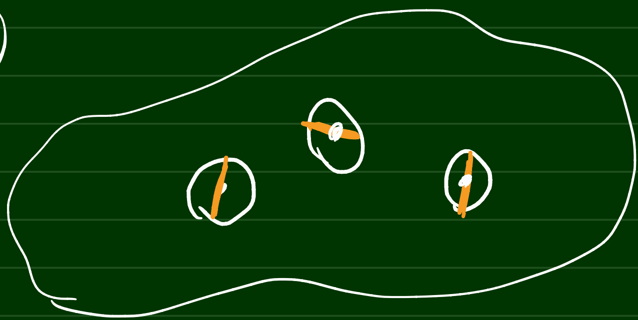
- λ : grading (line field) ← list of orientation ($\mathbb{Z}/2$-gr)

$\mathrel{\rotatebox{90}{$\in$}}$

section of $\mathbb{P}TS$

(up to homotopy で考える)





λがあるとき、$\mathbb{S}$上のarcに$\mathbb{Z}$-grを入れられる。

(λがないと $[2]=\mathrm{id}$ となる圏になる)

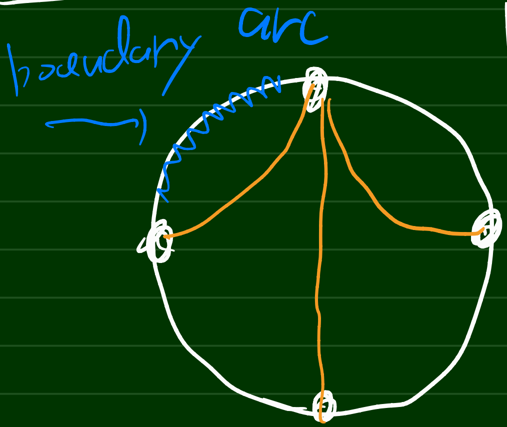
Def

$(\mathbb{S}, \mathbb{M}, \lambda)$ の full formal arc system

$T = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ とは

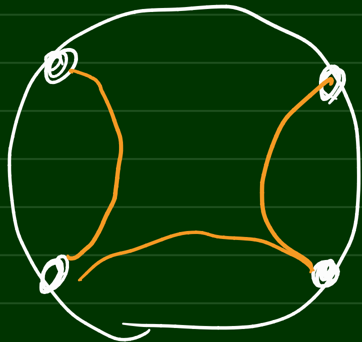
- γ_i : $\mathbb{M}$の点をつなぐ graded arc.
- $\mathbb{S} \setminus T$ は boundary arc を1つだけ含む polygon に分割されている $\square$

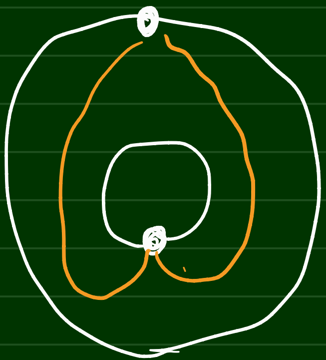
例



Koszul dual

$\longleftrightarrow$





Def

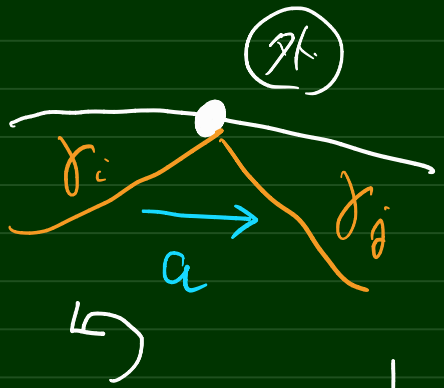
graded

$T \rightsquigarrow (Q_T^{(0)}, R_T)$: quiver with relation を

次のように与える.

- $Q_T^{(0)}$ の vertices $= \gamma_1, \gamma_2, \ldots, \gamma_n$

- 〃 arrow =

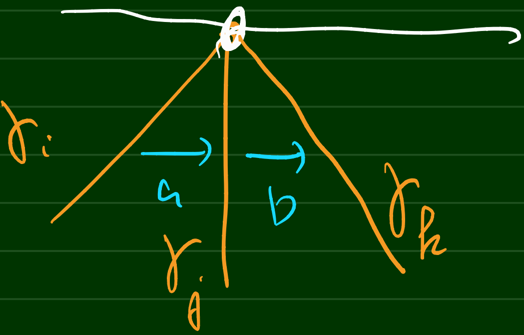


$i \xrightarrow{a} j$

$\deg a = \mathrm{index}(\gamma_i, \gamma_j) \in \mathbb{Z}$

- relation R_T

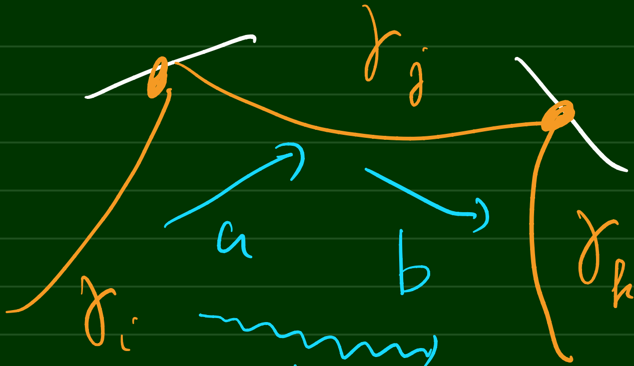
(1)



$$i \xrightarrow{a} j \xrightarrow{b} k$$

<u>no relation</u>

(2)



no intersection!

$$i \xrightarrow{a} j \xrightarrow{b} k$$

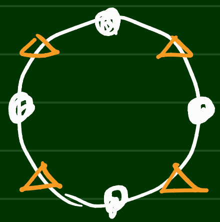
$$ab = 0.$$

$\Rightarrow\ A_T = Q^{(0)}_T / R_T$ is graded gentle alg $\square$

Dual FFAS

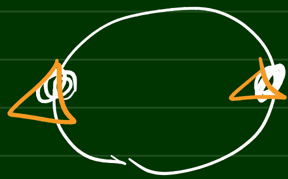
dual marked pt.

$(S, M) \rightsquigarrow (S, Y)$



Y : closed

M : open



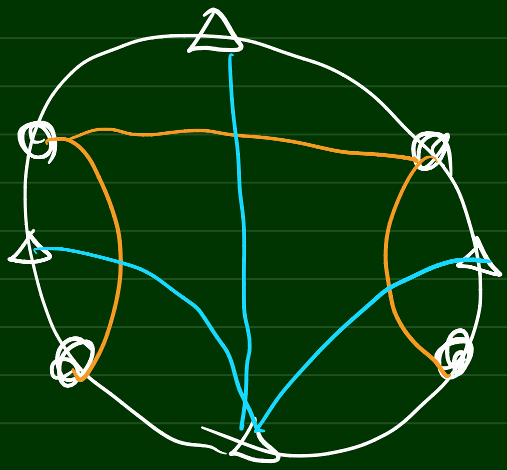
closed & [illegible] path $\rightsquigarrow$ cpt Lag

open " $\rightsquigarrow$ non-cpt Lag

T : FFAS of $(\mathbb{S}, \mathbb{M})$

$\updownarrow$

$T^\vee = \{\gamma_1^\vee, \ldots, \gamma_n^\vee\}$: FFAS of $(\mathbb{S}, \mathbb{Y})$



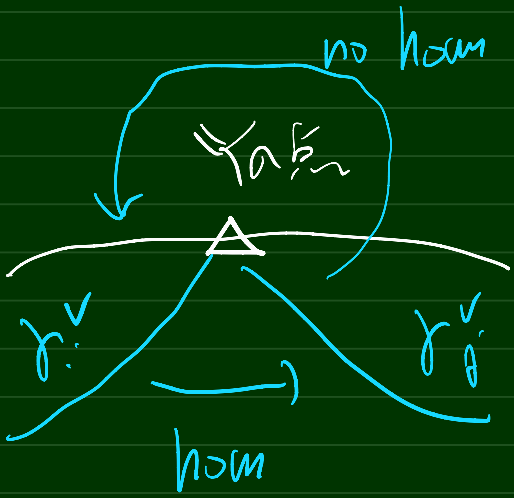
: 元の FFAS

: dual

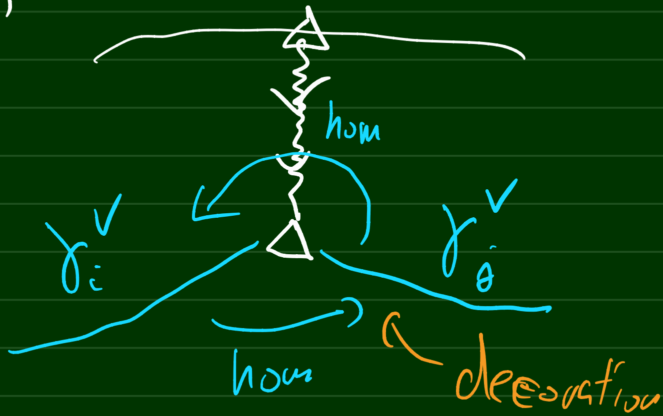
Koszul duality

$$\mathcal{O}_{fd}(A_T) \xrightarrow{\sim} \mathrm{Per}(A_{T^\vee})$$

$$\left(\begin{array}{c}\cong\\ \mathrm{Per}\end{array}\right) \qquad\qquad \left(\begin{array}{c}\cong\\ \mathcal{O}_{fd}\end{array}\right)$$



CY completion $\rightsquigarrow$



CY-X completion of gentle alg

dg resolution of $A_T = Q_T^{(0)} / R_T$

例

dg quiver

$$Q_T^{(0)} \; \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \Big/ ab=0 \quad \rightsquigarrow \quad Q_T^{(1)} \; \begin{tikzcd} \bullet \arrow[r, "a"] \arrow[rr, bend right, "c"'] & \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$$

$$dc = ab$$

$$\deg c = \deg a + \deg b - 1$$

例②

$$\begin{tikzcd} \bullet \arrow[r, "a_1"] & \bullet \arrow[r, "a_2"] & \bullet \arrow[r, "a_3"] & \bullet \end{tikzcd}$$

$$Q_T^{(0)} \Big/ \begin{matrix} a_1 a_2 = 0 \\ a_2 a_3 = 0 \end{matrix} \quad \rightsquigarrow$$

$$Q_T^{(1)} = \begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend right, "b_1"'] & \bullet \arrow[r] \arrow[rr, bend right, "b_2"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\Big\downarrow$$

$$Q_T^{(2)} = \begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend right, "b_1"'] \arrow[rrr, bend right=70, "c"'] & \bullet \arrow[r] \arrow[rr, bend right, "b_2"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$$\begin{cases} d b_1 = a_1 a_2 \\ d b_2 = a_2 a_3 \\ d c = \pm b_1 a_3 \pm a_1 b_2 \end{cases}$$

<u>Thm</u> (Oppermann)

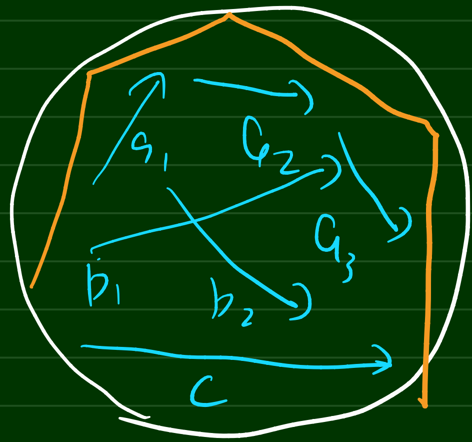
$\exists$ dg quiver Q_T

s.t. $(\mathbb{K}Q_T, d) \xrightarrow[\sim]{qis} \left(A_T = Q_T^{(0)}/R_T\right.$

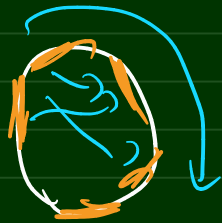
$\left., d=0\right)$

(d: Floer differential)

と解釈できる.

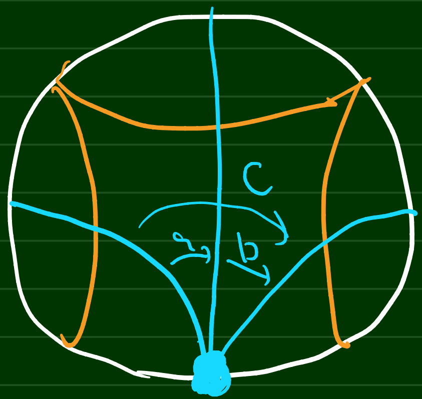


$Q_T^{(0)} \rightsquigarrow Q_T$ ← 各 polygon で



向きに全ての arrow をつけ足す。

Remark

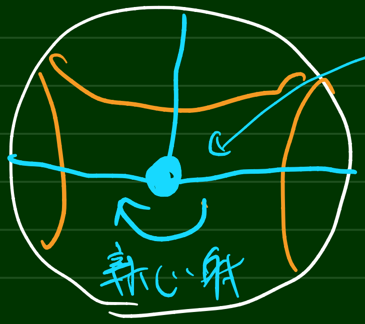


← Koszul dual.

τ const hol disk

⋮

{ CY completion.



ここの A_∞-relation $T^\vee$

Koszul dual

potential. T

CY completion

$\mathbb{Z}^2$-grading

$\cong$

$\mathbb{Z} \oplus \mathbb{Z} X$

dg resolution にする.

Q_T の double $\widetilde{Q}_T$ を

以下で作る.

• Q_Tの各 arrow $\bullet \xrightarrow{a} \bullet$ に対し、

opposite arrow $\bullet \xleftarrow{a^*} \bullet$ を $(2,-1)$

$$\deg a + \deg a^* = 2 - X$$

となるように入れる.

• loop
$$\begin{tikzcd} \underset{i}{\bullet} \arrow[loop above, "t_i"] \end{tikzcd}$$

$$\deg t_i = \cancel{3-X} \; 1 - X$$

の 2つを加える.

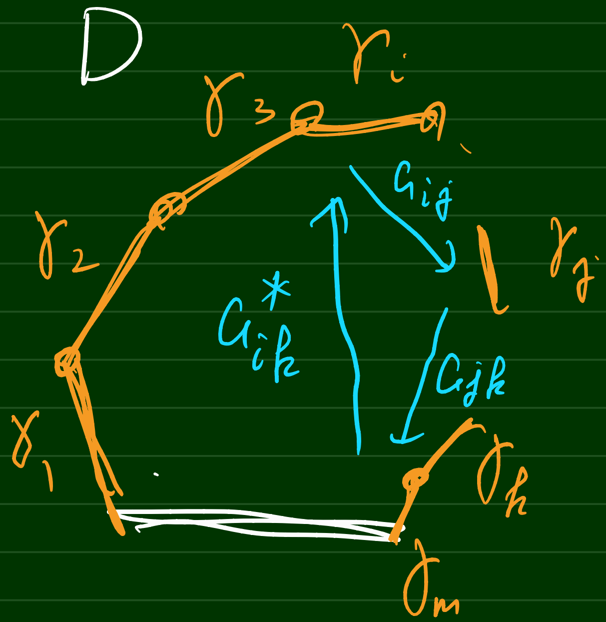
これに potential W_T を.

$$W_T = \sum W_D$$

D : S \ T の各 polygon

$$W_D := \sum_{1 \le i < j < k \le m} a_{ij}\, a_{jk}\, \underbrace{a_{ik}^{*}}_{1}$$

※ −gr を持っている



というように

Quiver with potential が得られる

⇝ Ginzburg dga $\quad \Gamma_T^{※} = (\widetilde{Q}_T, W_T)$

Thm (Keller)

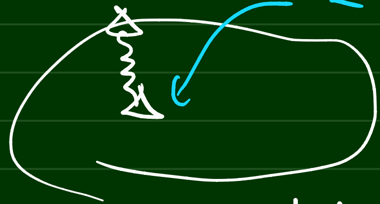
$\Gamma_T^{\mathbb{X}}$ は CY-$\mathbb{X}$ ← deformed CY-$\mathbb{X}$ completion of A_T i gr gentle alg.

Remark

($\mathcal{D}_{fd}(\Gamma_T^{\mathbb{X}})$ の stability cond = 多価の2次微分

I-Qiu

$z^{s-2}\,dz^{\otimes 2}$



$s \in \mathbb{C}$

の moduli

($s=3$ の CY-3 の場合の Bridgeland-Smith の $3 \rightsquigarrow s \in \mathbb{C}$ の)

一般化

4. Correspondence between arcs and spherical obj

- reachable sph obj.

$\widetilde{Q}_T$ の各頂点の simple modules $S_1, \cdots, S_n$ は

$\mathcal{D}_T^X$ の中で $S_1, \cdots, S_n$ は X-spherical.

$\left(\overset{\parallel}{\mathcal{D}_{fd}} (\Gamma_T^X) \right)$

$$\underbrace{ST(\mathcal{D}_T^X)}_{} = \langle \Phi_{S_1}, \cdots, \Phi_{S_n} \rangle$$

$$\Rightarrow \underbrace{ST(\mathcal{D}_T^X)\cdot\left\{ S_i[m+nX] \;\middle|\; \begin{matrix} m,n\in\mathbb{Z} \\ i=1,\dots,n \end{matrix} \right\}}_{\text{reachable spherical obj}}$$

$$\parallel \; \underline{\text{conj}}$$

all spherical.

eg DMS

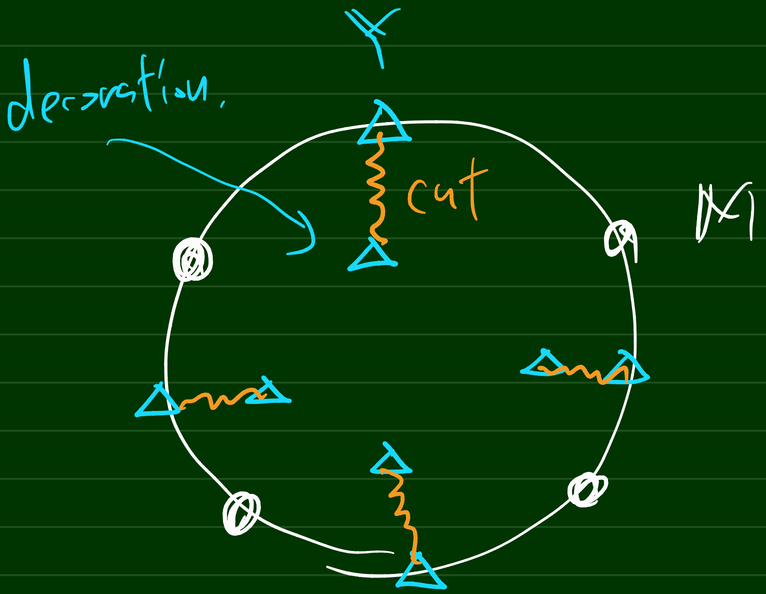
- decoration Δ on S

$$\text{def} \iff \underset{\text{finite pts}}{\Delta} \subset S \setminus \partial S$$

$$\begin{matrix} m \\ \parallel \\ |\Delta| \end{matrix} = |Y| = |M|$$

• cut $C = \{c_1, c_2, \cdots, c_m\}$

$\overset{\text{def}}{\Longleftrightarrow}$ △とYの点と1つずつつなぐ pathの homotopy 類、

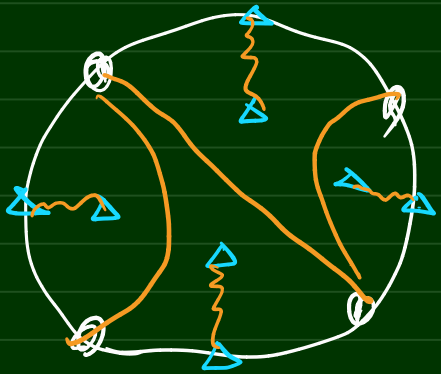


• CがTとcompatible $\overset{\text{def}}{\Longleftrightarrow}$ CとTが交わらない.

〜〜〜 ① T をとる.

② 各 polygon に 1つ decoration をとって $\triangle$ を作る.

③ ユニークに compatible cut が決まる.



$\mathbb{S}_\triangle$ に cut が与えられてる時.

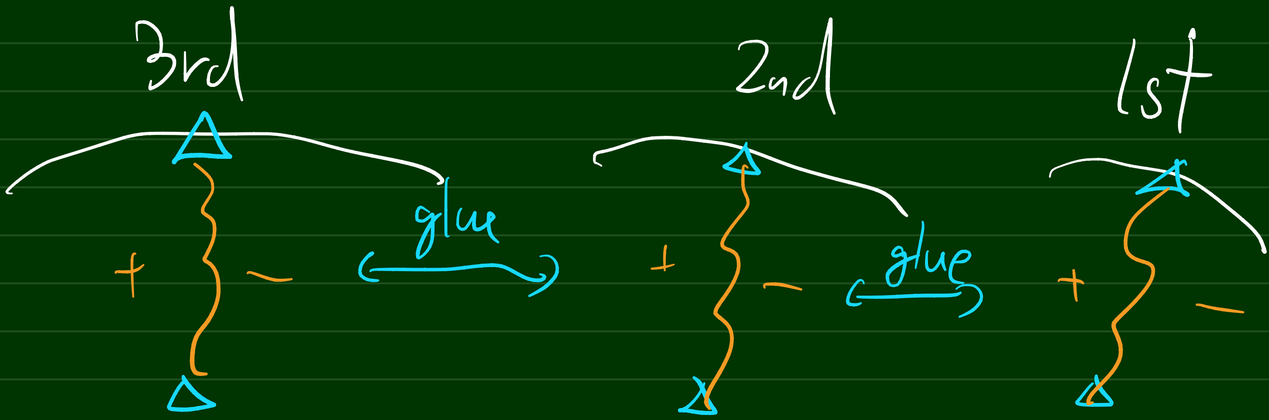
$\|$

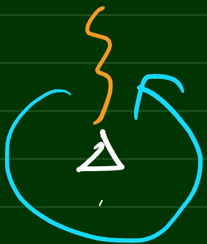
$(\mathbb{S}, \triangle)$

cut に沿って Z 個の $\mathbb{S}_\triangle$ を

貼り合わせて作った $\mathbb{S} \setminus \triangle$ の

infinite cyclic cover と $\log \mathfrak{S}_{\Delta}$.





1周するとシートが変わる.

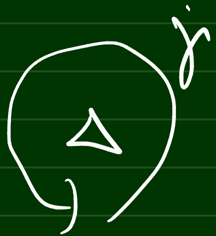
⇕

χ-grading shift.

$\lambda \in H^1(\mathbb{P}T\mathbb{S};\mathbb{Z})$

$\rightsquigarrow$ lift.

$\Lambda \in H^1(\mathbb{P}T(\mathbb{S}\setminus\Delta);\mathbb{Z}\oplus\mathbb{Z}\mathbb{X})$

s.t. 

$\langle \Lambda, \gamma \rangle = 2 - \mathbb{X}$.

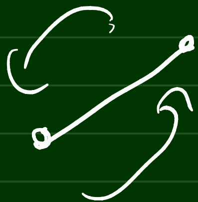
$\log \mathbb{S}_\Delta \pm 1 := \Lambda$: $\mathbb{Z}^2$-grading data.

<u>Thm</u> (I-Qiu-Zhou)

l_g S_Δの Δの点をつなぐ

$\mathbb{Z}^2$-graded arcs $\overset{1:1}{\longleftrightarrow}$ reachable sph obj

(+ condition)



arcに沿った twist $\longleftrightarrow$ spherical twist

$\cap$ $\cap$

(mapping class grp) cat の autoequiv.

$$\mathrm{Per}_{\mathrm{IS}}\, A_T$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{D}_{fd}(\Gamma^{\mathbb{X}}_{T}) \arrow[r] \arrow[d, "\mathbb{X}=N"] & \mathrm{Per}(\Gamma^{\mathbb{X}}_{T}) \arrow[r] \arrow[d, equal, "\mathbb{X}=N"] & \mathcal{C}(\Gamma^{\mathbb{X}}_{T}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{D}_{fd}(\Gamma^{N}_{T}) \arrow[r] & \mathrm{Per}(\Gamma^{N}_{T}) \arrow[r] & \mathcal{C}(\Gamma^{N}_{T}) \arrow[r] & 0
\end{tikzcd}$$