Monoidal Abelian Envelopes

Tokyo - Nagoya Algebra Seminar

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Fix a field k.

A k-linear symmetric monoidal category $(T, \otimes, \mathbb{1})$ is a **tensor category** over k if

- T is abelian
- $(T, \otimes, \mathbb{1})$ is rigid (for $X \in T$, we have dual $X^* \in T$)
- $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism

linear symmetric monoidal

Exact tensor functors between tensor categories are faithful.

Examples

① $\mathrm{Rep}_{\Bbbk} G$ for a "group G"

- G finite group
- G top group, $\Bbbk = \mathbb{R}$, continuous representations
- G algebraic group, algebraic representations

② $\mathrm{Sh}(X;\Bbbk)^{\text{loc. const.}}$ for nice connected topological space X

$\cong \mathrm{Rep}_{\Bbbk}\, \pi_1(X, x_0)$

③ $\mathrm{sRep}\, G$ for algebraic supergroup

e.g. $G = GL(m|n)$

Principle "tensor categories like to be Rep G
for an affine group scheme G"

From now on $k = \bar{k}$.

Theorem (Deligne '90)

For a tensor category T/k with char $k = 0$

- $T \cong \mathrm{Rep}_k G$ for an affine group scheme G

$\Updownarrow$

- $\forall X \in T$, $\Lambda^\bullet X$ is finite

$\exists$ Recent results of this type for char $k = p > 0$

(C., Etingof, Gelaki, Ostrik)

They involve "Frobenius twist"

Example: For $X \in T$, define $\mathrm{Fr}\, X$ as image of

$$H^0(S_p, \otimes^p X) \hookrightarrow \otimes^p X \twoheadrightarrow H_0(S_p, \otimes^p X)$$

$X \mapsto \mathrm{Fr}\, X$ is additive, even $\mathbb{F}_p$-linear.

Is Fr exact? (Q1) Known not the case for $p=2$.

Back to char $k=0$

For $\delta \in k$, is there a "universal tensor category on one object of categorical dimension δ"?

$\left(\dim X : \mathbb{1} \xrightarrow{co} X \otimes X^* \xrightarrow{ev} \mathbb{1}\right)$

That is, can we construct

- tensor category U_δ
- $X_\delta \in U_\delta$

with

$$\begin{aligned} F &\longmapsto F(X_\delta) \\ \mathrm{Tens}^{ex}(U_\delta, T) &\longrightarrow \{\text{objects in } T \text{ of dimension } \delta\} \end{aligned} \quad ?$$

No!

Rep $GL(n)$ $\quad V = k^n$

$$\Lambda^{n+1} X_n = 0 \quad \Longleftarrow \quad \Lambda^{n+1} V = 0$$

Rep $GL(n+1|1)$ $\quad V = k^{n+1|1}$

$$\Lambda^{n+1} X_n \neq 0 \quad \Longleftarrow \quad \Lambda^{n+1} V \neq 0$$

Contradiction.

$\rightsquigarrow$ Do we have a collection $\{U_\delta^\alpha\}$ which together "classify objects"? $\quad$ (Q2)

A k-linear symmetric monoidal category $(D, \otimes, \mathbb{1})$ is a pseudo-tensor category over k if

- D is pseudo-abelian
- $k \to \mathrm{End}(\mathbb{1})$ is an isomorphism
- $(D, \otimes, \mathbb{1})$ is rigid

= additive × idempotent complete

"get for free"

Examples

(1) Easy to construct
- diagrammatically
- universal property
- generators and relations

Example $\mathcal{OB}(\delta)$ (oriented Brauer category)

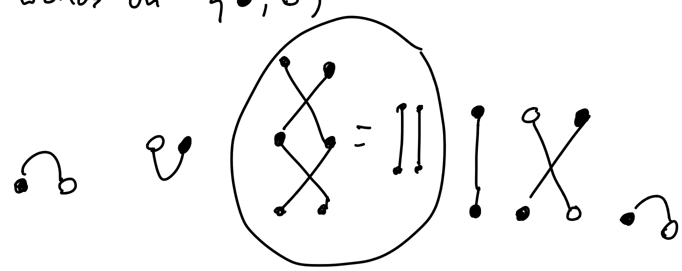
a pseudo-tensor category with object X_δ

$$\mathrm{Tens}(\mathcal{OB}(\delta), \mathcal{D}) \xrightarrow{\sim} \{\text{objects in } \mathcal{D} \text{ of dimension } \delta\}$$
$$F \longmapsto F(X_\delta)$$

It is the pseudo-abelian envelope of $\mathcal{OB}(\delta)_0$

with $\mathcal{OB}(\delta)_0$ = words in $\{\bullet, \circ\}$

morphisms:



(2) Tilt G for G reductive

Example Tilt $SL(2)$ = category of direct summands of direct sums of $\otimes^i V$

$(V = k^2)$

$(\subset \mathrm{Rep}\, SL(2))$

(full)

Definition

A faithful tensor functor $F: D \to T$ from a pseudo-tensor category into a tensor category is an **abelian envelope** if for each tensor category T'

$$\mathrm{Tens}^{ex}(T, T') \xrightarrow{\cong} \mathrm{Tens}^{faith}(D, T')$$

(D admits an abelian envelope iff 2-functor

$$\mathrm{Tens}^{faith}(D, -) : \left(\begin{array}{c}\text{tensor cat} \\ \text{exact tensor functor} \\ \text{monoidal nat transfo}\end{array}\right) \longrightarrow \mathrm{Cat}$$

is representable.)

Examples

- Comes - Ostrik
- Entova - Hinich - Serganova $(/\mathbb{C})$

Abelian envelope of $\mathcal{GB}(\delta)$, $\delta \in \mathbb{Z}$.

Idea: For $sRep\, GL(a|b)$, $a-b=\delta$

Subcategory of subquotients of direct sums of $V^{\otimes i} \otimes (V^*)^{\otimes j}$ $i+j \leq n$

does not depend on a,b if $n << a,b$

$$\mathcal{GB}^{ab}(\delta) = \varinjlim_{n\to\infty} \Big(\varprojlim_{\substack{a,b\to\infty \\ a-b=\delta}} sRep\, GL(a|b)^{\leq n} \Big)$$

Classification tensor ideals (C.)

$\mathcal{GB}(\delta) \supset I_0 \supset I_1 \supset I_2 \supset I_3 \supset \cdots$

Abelian envelope of $(\delta > 0)$

$\mathcal{GB}(\delta)/I_i$ is $\mathrm{Rep}\, GL(\delta+i \mid i)$

$\Rightarrow \left\{ \mathrm{Rep}\, GL(\delta+i \mid i), \mathcal{GB}^{ab}(\delta) \right\}$

is a "collection of universal tensor categories"

$\boxed{Q2}$

- Tensor ideals in $\mathrm{Tilt}\, SL(2)$

$\mathrm{Tilt}\, SL(2) \supset I_0 \supset I_1 \supset I_2 \supset \cdots$

$(\mathrm{char}\, k = p > 0)$

Benson - Etingof

If $p = 2$ $\quad$ $\mathrm{Tilt}\, SL(2)/I_i$ have abelian envelopes

On which F_n is not exact!

- C.-Entova-Heidersdorf
  for general reductive G and char k
  $\mathrm{Rep}\, G$ is abelian envelope of $\mathrm{Tilt}\, G$

  Moreover, as conjectured by Benson-Etingof,
  for $p=2$

$$\mathrm{Rep}\, SL(2) \cong \varinjlim_{n\to\infty} \varprojlim_{i\to\infty} \left(\left(\mathrm{Tilt}\, SL(2)/I_i\right)^{ab}\right)^{\leq 2}$$

Can we know that $\mathcal{D}$ admits an abelian envelope, without having it already?

Theorem (C.) Let $\mathcal{D}$ be a pseudo-tensor category.
Assume that for every $f: A \to B$ in $\mathcal{D}$

there is $X \in D$ for which

* $X \otimes f$ is split
* $X^* \otimes X \otimes X^* \otimes X \underset{X^* \otimes X \otimes ev}{\overset{ev \otimes X^* \otimes X}{\rightrightarrows}} X^* \otimes X \xrightarrow{ev} \mathbb{1}$ is a coequaliser

Then D admits an abelian envelope T and $\operatorname{Ind} T \cong Sh(D, \tau)$ for some Grothendieck topology on D.

Concretely
$F \in [D^{op}, Ab]$ is a sheaf iff

$$F(A) \to F(A \otimes X^* \otimes X) \rightrightarrows F(A \otimes X^* \otimes X \otimes X^* \otimes X)$$

is an equaliser

$\forall A, X$

Applicable to all above examples as well as to $\operatorname{Tilt} SL_2 / I_i$ for $p > 2$.

Simultaneously Benson - Etingof - Ostrik

abelian envelopes of Tilt SL_2/I_i for $p>2$

$\to$ They show Fr is not exact! $\boxed{Q1}$

Theorem (C.) For a pseudo-tensor category D and embedding $D \hookrightarrow T$, such that every $X \in T$ is a quotient of some $Y \in D$, then $D \hookrightarrow T$ is an abelian envelope.

Every <u>known</u> abelian envelope is of the above type.

Corollary Every tensor category is its own abelian envelope.

Theorem (C. - Etingof - Ostrik - Pauwels)

If $D \subset \mathrm{Rep}\, G$, then D has an abelian envelope $\mathrm{Rep}\, H$ and $D \subset \mathrm{Rep}\, H$ has the quotient property.

Conjecture $D \subset T$ is an abelian envelope iff we have the quotient property.