ICE-closed subcategories and wide τ-tilting modules

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§0 Intro

In rep. theory of f.d. alg.

there are many results of the forms

subcat. of modΛ $\overset{1-1}{\longleftrightarrow}$ obj. in modΛ

e.g. [Adachi-Iyama-Reiten]

$$
\begin{array}{ccc}
f\text{-tors}\Lambda & \overset{1-1}{\longleftrightarrow} & s\tau\text{-tilt}\Lambda \\
\cap & & \cap \\
\text{df-ice}\Lambda & \overset{1-1}{\longleftrightarrow} & w\tau\text{-tilt}\Lambda
\end{array}
$$

<u>Today</u>: the second row.

<u>Setting</u>

k : field Λ : f.d. k-alg.

modΛ : the cat. of f.g. left Λ-modules.

$\mathcal{A}$: an abelian length cat. (e.g. modΛ)

For $\mathcal{C} \subset \mathcal{A}$: subcat.

$\mathcal{C}^{\perp} := \{A \in \mathcal{A} \mid \mathrm{Hom}(C, A) = 0 \ {}^{\forall}C \in \mathcal{C}\}$

${}^{\perp}\mathcal{C} := \{ \text{———} (A, C) \text{———} \}$

§1 Torsion classes and wide subcat.

<u>Def.</u> $\mathcal{C} \subset \mathcal{A}$: subcat.

$\mathcal{C}$ is closed under

(1) extensions

$:\Leftrightarrow {}^{\forall}(0 \to L \to M \to N \to 0)$: s.e.s. in $\mathcal{A}$

$L, N \in \mathcal{C} \Rightarrow M \in \mathcal{C}$

(2) sub

$:\Leftrightarrow$ If $M \in \mathcal{C}$, then $L \in \mathcal{C}$.

(3) quotients

$:\Leftrightarrow$ If $M \in \mathcal{C}$, then $N \in \mathcal{C}$.

(4) cokernels

$:\Leftrightarrow {}^{\forall}f : X \to Y$ in $\mathcal{C}$

$\mathrm{Cok}\, f \in \mathcal{C}$

(5) kernels

$:\Leftrightarrow \mathrm{Ker}\, f \in \mathcal{C}$

(6) images

$:\Leftrightarrow \mathrm{Im}\, f \in \mathcal{C}$

Def. $\mathcal{T} \subset \mathcal{A}$: subcat.

$\mathcal{T}$ is a torsion(-free) class

$:\Leftrightarrow$ closed under (tors. torf.)

ext. and quot. (sub.)

$\operatorname{tors}\mathcal{A}$: the set of tors. in $\mathcal{A}$

$\operatorname{torf}\mathcal{A}$: ―――― torf. ――

Rmk. Since $\mathcal{A}$: length

$\mathcal{T}$: tors. $\Leftrightarrow$ $(\mathcal{T}, \mathcal{T}^{\perp})$: a tors. pair in $\mathcal{A}$

e.g. $\{\text{torsion grp}\} \subset \operatorname{Mod}\mathbb{Z}$: tors.

e.g. $Q : 1 \to 2 \leftarrow 3$

kQ : path alg.

modkQ

$$\begin{tikzcd}[column sep=small, row sep=small]
 & {\begin{smallmatrix}1\\2\end{smallmatrix}} \arrow[dr] \arrow[rr, dash, dashed] & & 3 & \\
2 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & {\begin{smallmatrix}1\,3\\2\end{smallmatrix}} \arrow[ur] \arrow[dr] & & \\
 & {\begin{smallmatrix}3\\2\end{smallmatrix}} \arrow[ur] \arrow[rr, dash, dashed] & & 1 &
\end{tikzcd}$$

: AR quiver of modkQ

● : tors.

Fact.

$$\operatorname{tors}\mathcal{A} \underset{{}^{\perp}(-)}{\overset{(-)^{\perp}}{\rightleftarrows}} \operatorname{torf}\mathcal{A}$$

: inc. reversing bij.

Def. $\mathcal{W} \subset \mathcal{A}$: subcat.

$\mathcal{W}$ is a wide subcat.

:$\Longleftrightarrow$ closed under

ext. cok. and ker.

Fact. $\mathcal{W}$ is also

an abelian length cat.

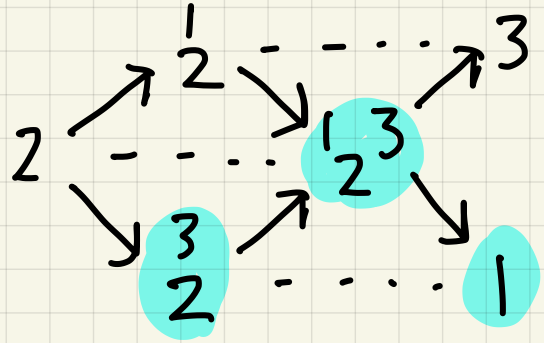
e.g. Serre subcat.

(:$\Longleftrightarrow$ closed under

ext. sub. and quot.)

is wide.

e.g.



: wide

not Serre

§2 ICE-closed subcat.

Def. $\mathcal{C} \subset \mathcal{A}$: subcat.

$\mathcal{C}$ is an ICE-closed subcat.

:$\Longleftrightarrow$ closed under (ICE)

Images, Cok. and Ext.

e.g. tors. and wide are ICE-closed

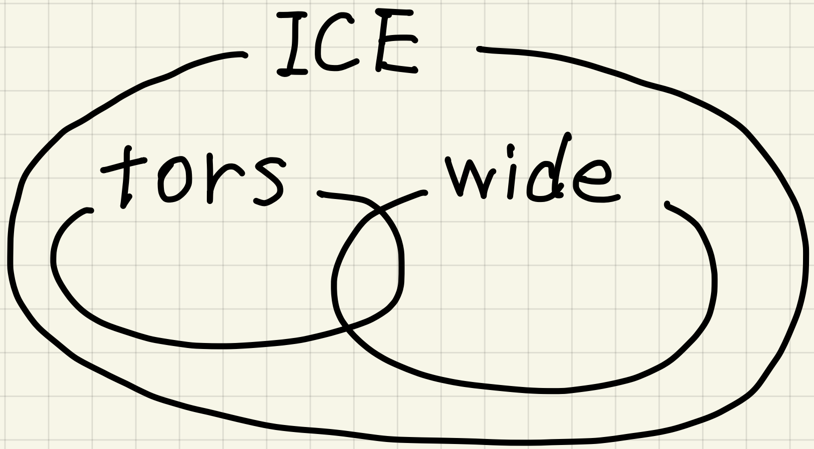
$\because$ ∘ ext. : O.K.

∘ $X \xrightarrow{f} Y$ in $\mathcal{C}$

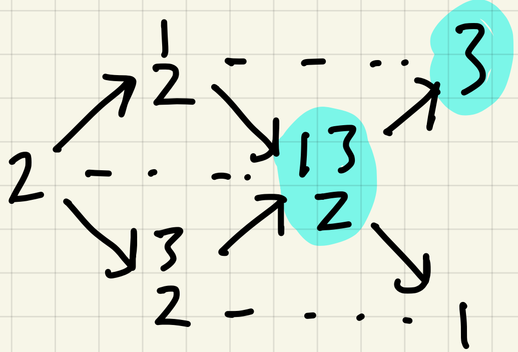
$$\begin{tikzcd} X \arrow[dr] \arrow[rr, "f"] & & Y \arrow[dr] & \\ & \mathrm{Im} f \arrow[ur] & & \mathrm{Cok} f \end{tikzcd}$$

If $\mathcal{C}$: tors.
then $\mathrm{Im} f$, $\mathrm{Cok} f \in \mathcal{C}$ ($\because$ quot.)

If $\mathcal{C}$: wide,
then $\mathrm{cok} f \in \mathcal{C}$
and $\mathrm{Im} f = \mathrm{Ker}(Y \to \mathrm{Cok} f) \in \mathcal{C}$ $\lrcorner$



e.g.



▇ : ICE
not tors.
not wide.

Prop. 1 $\mathcal{W} \subset \mathcal{A}$: wide

$\mathcal{T} \subset \mathcal{W}$: tors.

($\mathcal{W}$: viewed as abelian)

Then $\mathcal{T} \subset \mathcal{A}$: ICE

Proof. ∘ ext. : O.K.

∘ $X \xrightarrow{f} Y$ in $\mathcal{T}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dr] & \\ & \mathrm{Im} f \arrow[ur] & & \mathrm{Cok} f \end{tikzcd}$$

Since $\mathcal{W}$: wide

$\mathrm{Im} f, \mathrm{Cok} f \in \mathcal{W}$

Since $\mathcal{T} \subset \mathcal{W}$: tors.

$\mathrm{Im} f, \mathrm{Cok} f \in \mathcal{T}$ □

The converse holds

in a sense.

Thm. 2 [Enomoto - S]

$\mathcal{T} \subset \mathcal{A}$: subcat.

TFAE

(1) $\mathcal{T} \subset \mathcal{A}$: ICE

(2) $^\exists \mathcal{W} \subset \mathcal{A}$: wide

s.t. $\mathcal{T} \subset \mathcal{W}$: tors.

Proof overview

We make use of

intervals in $\mathrm{tors}\,\mathcal{A}$

Def. $\mathcal{T}, \mathcal{U} \in \operatorname{tors} \mathcal{A}$, $\mathcal{U} \subset \mathcal{T}$

$[\mathcal{U}, \mathcal{T}] := \{\mathcal{T}' \in \operatorname{tors} \mathcal{A} \mid \mathcal{U} \subset \mathcal{T}' \subset \mathcal{T}\}$

: an interval in $\operatorname{tors} \mathcal{A}$

$\mathcal{H}[\mathcal{U}, \mathcal{T}] := \mathcal{T} \cap \mathcal{U}^{\perp} \subset \mathcal{A}$

: the heart of $[\mathcal{U}, \mathcal{T}]$

"the difference between $\mathcal{U}$ and $\mathcal{T}$"

- $\mathcal{H}[\mathcal{T}, \mathcal{T}] = \mathcal{T} \cap \mathcal{T}^{\perp} = 0$
- $\mathcal{H}[0, \mathcal{T}] = \mathcal{T} \cap 0^{\perp} = \mathcal{T}$

Prop. 3 [ES]

$\mathcal{C} \subset \mathcal{A}$: ICE

Then ${}^{\exists}[\mathcal{U}, \mathcal{T}] \subset \operatorname{tors} \mathcal{A}$: int.

s.t. $\mathcal{C} = \mathcal{H}[\mathcal{U}, \mathcal{T}]$

Prop. 4 [Asai-Pfeifer]

$[\mathcal{U}, \mathcal{T}] \subset \operatorname{tors} \mathcal{A}$: wide int.

i.e. $\mathcal{H}[\mathcal{U}, \mathcal{T}] \subset \mathcal{A}$: wide

Then $\varphi := (-) \cap \mathcal{U}^{\perp}$

$[\mathcal{U}, \mathcal{T}] \xrightarrow{\sim} \operatorname{tors} \mathcal{H}[\mathcal{U}, \mathcal{T}]$

: bij.

Moreover $[\mathcal{U}',\mathcal{T}']\subset[\mathcal{U},\mathcal{T}]$

$\Rightarrow \mathcal{H}[\mathcal{U}',\mathcal{T}'] = \mathcal{H}[\varphi(\mathcal{U}'),\varphi(\mathcal{T}')]$

$$\xrightarrow{\varphi}$$

$$\begin{tikzcd}
\mathcal{T} \arrow[rr] \arrow[d, dash] & & \mathcal{H}[\mathcal{U},\mathcal{T}] \arrow[d, dash] \\
\mathcal{T}' \arrow[rr] \arrow[d, dash] & & \varphi(\mathcal{T}') \arrow[d, dash] \\
\mathcal{U}' \arrow[rr] \arrow[d, dash] & & \varphi(\mathcal{U}') \arrow[d, dash] \\
\mathcal{U} \arrow[rr] & & 0
\end{tikzcd}$$

$\mathcal{H}\,[\mathcal{U}',\mathcal{T}']$ $\qquad$ $[\varphi(\mathcal{U}'),\varphi(\mathcal{T}')]\,\mathcal{H}$

$[\mathcal{U},\mathcal{T}]$ $\qquad\qquad$ $\operatorname{tors}\mathcal{H}[\mathcal{U},\mathcal{T}]$

Thm. 5 [ES]

$[\mathcal{U},\mathcal{T}]\subset \operatorname{tors}\mathcal{A}$: int.

TFAE

(1) $[\mathcal{U},\mathcal{T}]$: ICE int.

i.e. $\mathcal{H}[\mathcal{U},\mathcal{T}]\subset\mathcal{A}$: ICE

(2) $\exists\, \mathcal{T}'\in\operatorname{tors}\mathcal{A}$

s.t. $\mathcal{T}\subset\mathcal{T}'$ and

$[\mathcal{U},\mathcal{T}']$: wide int.

In this case,

$\mathcal{H}[\mathcal{U},\mathcal{T}]\subset\mathcal{H}[\mathcal{U},\mathcal{T}']$: tors.

tors $\mathcal{A}$

$$\begin{tikzcd}
 & \mathcal{T}' \arrow[dl] \arrow[dr] & \\
\cdot \arrow[d, dash, dotted] & & \mathcal{T} \arrow[d] \\
\cdot \arrow[dr] & & \cdot \arrow[dl] \\
 & \mathcal{U} &
\end{tikzcd}$$

: ICE int.

$[\mathcal{U},\mathcal{T}']$: wide int.

Sketch of proof of (2) $\Rightarrow$ (1)

$$\begin{tikzcd}
\mathcal{T}' \arrow[r, "\varphi"] \arrow[d, dash] & \mathcal{W} := \mathcal{H}[\mathcal{U},\mathcal{T}'] \arrow[d, dash] \\
\mathcal{T} \arrow[r] \arrow[d, dash] & \varphi(\mathcal{T}) \arrow[d, dash] \\
\mathcal{U} \arrow[r] & 0
\end{tikzcd}$$

$\mathcal{H}[\ \ $ $[\mathcal{U},\mathcal{T}']$ $\quad$ $\operatorname{tors}\mathcal{W}$

$$\mathcal{H}[\mathcal{U},\mathcal{T}] \overset{\text{Prop. 4}}{=} \mathcal{H}([0,\varphi(\mathcal{T})])$$

$$= \varphi(\mathcal{T}) \cap \mathcal{W} \subset \underset{\text{tors.}}{\mathcal{W}}$$

By prop. 1 $\mathcal{H}[\mathcal{U},\mathcal{T}] \subset \mathcal{A}$: ICE ┘

Proof of Thm. 2

$\mathcal{C} \subset \mathcal{A}$: ICE

By prop. 3, $\exists [\mathcal{U},\mathcal{T}] \subset \operatorname{tors}\mathcal{A}$

s.t. $\mathcal{C} = \mathcal{H}[\mathcal{U},\mathcal{T}]$

By thm. 5, $\exists \mathcal{T}' \in \operatorname{tors}\mathcal{A}$,

$\mathcal{C} = \mathcal{H}[\mathcal{U},\mathcal{T}] \subset \mathcal{H}[\mathcal{U},\mathcal{T}']$: wide

tors. □

§3 Wide τ-tilting modules

In the rest, $\mathcal{A} := \mathrm{mod}\,\Lambda$

Recall [AIR]

$$\mathrm{f\text{-}tors}\,\Lambda \overset{1-1}{\longleftrightarrow} \mathrm{s}\tau\text{-}\mathrm{tilt}\,\Lambda$$

<u>Aim</u> Extend this bijection.

<u>Def.</u> $\mathcal{C} \subset \mathrm{mod}\,\Lambda$: ICE

is doubly functorially finite (d.f.f.)

$:\Leftrightarrow {}^{\exists}\mathcal{W} \subset \mathcal{A}$: f.f. wide

s.t. $\mathcal{C} \subset \mathcal{W}$: f.f. tors.

$\mathrm{df\text{-}ice}\,\Lambda$: the set of

d.f.f. ICE of $\mathrm{mod}\,\Lambda$

$$\mathrm{f\text{-}tors}\,\Lambda \subset \mathrm{df\text{-}ice}\,\Lambda$$

<u>Fact.</u> $\mathcal{W} \subset \mathrm{mod}\,\Lambda$: wide

$\mathcal{W}$: f.f. $\Leftrightarrow {}^{\exists}\Gamma$: f.d. k-alg.

s.t. $\mathcal{W} \simeq \mathrm{mod}\,\Gamma$

<u>Recall</u> $T \in \mathrm{mod}\,\Lambda$

is supp. τ-tilt. module

$:\Leftrightarrow {}^{\exists}e \in \Lambda$: idempotent

s.t. $T \in \mathrm{mod}\,\Lambda/\langle e\rangle$: τ-tilt.

$\Leftrightarrow {}^{\exists}\mathcal{S} \subset \mathrm{mod}\Lambda$: Serre subcat.

s.t. $T \in \mathcal{S}$: τ-tilting.

($\mathcal{S}$: viewed as $\mathrm{mod}\Lambda/\langle e\rangle$)

Def. $T \in \mathrm{mod}\Lambda$

is wide τ-tilting module

$:\Leftrightarrow {}^{\exists}\mathcal{W} \subset \mathrm{mod}\Lambda$: f.f. wide

s.t. $T \in \mathcal{W}$: $\tau_{\mathcal{W}}$-tilting

($\mathcal{W}$: viewed as $\mathrm{mod}\,{}^{\exists}\Gamma$)

$\mathrm{w}\tau\text{-tilt}\Lambda$: the set of

iso-classes of basic

wide τ-tilting modules.

Thm. 6 [ES]

There are bijections.

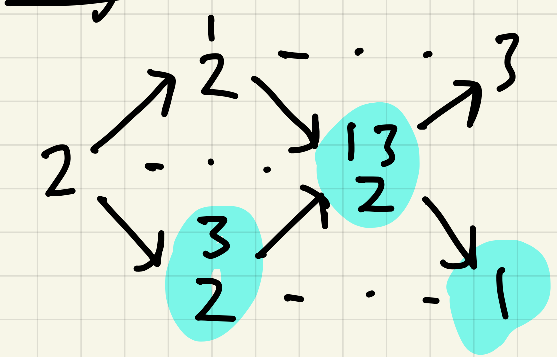
$$\begin{array}{ccc} \mathrm{w}\tau\text{-tilt}\Lambda & \underset{P(-)}{\overset{\mathrm{cok}(-)}{\rightleftarrows}} & \mathrm{df}\text{-}\mathrm{ice}\Lambda \\ \cup & & \cup \\ \mathrm{s}\tau\text{-tilt}\Lambda & \underset{[\mathrm{AIR}]}{\rightleftarrows} & \mathrm{f}\text{-}\mathrm{tors}\Lambda \end{array}$$

$\mathrm{cok}\,T$: the cat. consisting

of cokernels of

maps in $\mathrm{add}\,T$.

e.g.



: ICE

$\begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\,3\\2\end{smallmatrix}$: wide τ-tilt.

not supp. τ-tilt.

we want to study

$\vec{H}(\text{df-ice}\Lambda)$: Hasse quiver

In the rest,

Q : Dynkin $\Lambda := kQ$

Thm. 7 [ES]

(1) $T \in \operatorname{mod}\Lambda$

T : wide τ-tilt.

$\Leftrightarrow$ T : rigid i.e. $\operatorname{Ext}^1_\Lambda(T,T)=0$

(2) [Enomoto]

$$\operatorname{rigid}\Lambda \xleftrightarrow{1-1} \operatorname{ice}\Lambda$$

$\operatorname{rigid}\Lambda$: the set of iso-classes of basic rigid modules.

We identify

$$\vec{H}(\mathrm{rigid}\,\Lambda) = \vec{H}(\mathrm{ice}\,\Lambda)$$

Thm. 8 $T \in \mathrm{mod}\,\Lambda$: rigid

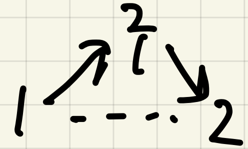
(1) ${}^{\forall}X$: indec. summand of T

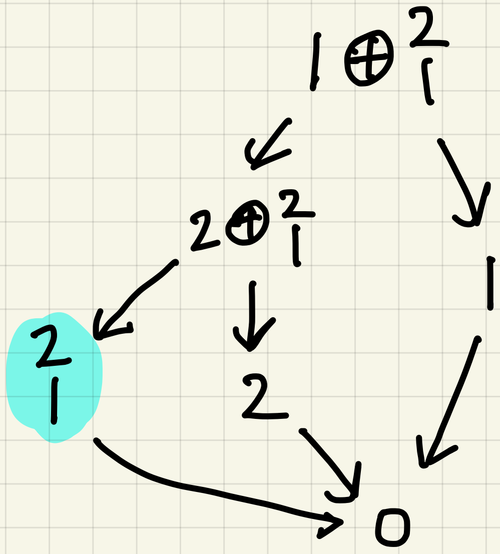
${}^{\exists}\mu_X(T)$: rigid s.t.

$$T \longrightarrow \mu_X(T) \text{ in } \vec{H}(\mathrm{rigid}\,\Lambda)$$

(2) Every arrow in $\vec{H}(\mathrm{rigid}\,\Lambda)$ is of this form.

e.g. $Q : 1 \longleftarrow 2$





$$\vec{H}(\mathrm{rigid}\,kQ)$$

: rigid

not $s\tau$-tilt.