Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2023 Admission

Part 2 of 2

February 5, 2023, 9:00 〜12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let X and Y be real matrices that are antisymmetric (i.e., ${}^tX = -X$ and ${}^tY = -Y$, where ${}^tX$ and ${}^tY$ denote the transposes of X and Y). Answer the following questions.

(1) Show that $(XY - YX)^2$ is a real symmetric matrix.

(2) Show that $\mathrm{tr}(XY - YX)^4 \geq 0$.

(3) Show that X and Y commute if and only if equality in (2) holds.

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2 Using a sequence of real numbers $\{a_k\}$ satisfying $\sum_{k=1}^{\infty} |a_k| < \infty$, we define the function $u(x,t)$ on $\mathbb{R} \times [0,\infty)$ as follows.

$$u(x,t) = \sum_{k=1}^{\infty} a_k (\sin kx) e^{-k^2 t} \qquad (x \in \mathbb{R},\ t \geq 0)$$

Answer the following questions.

(1) Show that $u(x,t)$ is a continuous function on $\mathbb{R} \times [0,\infty)$.

(2) Show that $u(x,t)$ converges uniformly on $\mathbb{R}$ to $u(x,0)$ when $t \to 0$.

(3) Show that $u(x,t)$ is of class C^1 on $\mathbb{R} \times (0,\infty)$.

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3 For $0 < \alpha < 2$, define the complex function f as $f(z) = \dfrac{z^{\alpha-1}}{1+z^2}$. Answer the following questions.

(1) For $z = re^{i\theta}$ $(r > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2})$, express $z^{\alpha-1}$ using r and θ.

(2) Let D be the domain obtained by removing from the complex plane $\mathbb{C}$ the origin and the imaginary axis in the lower-half plane. Based on (1), we consider f as a single-valued function defined on D. For $0 < \varepsilon < 1 < R$, we define the closed curve

$$C(\varepsilon, R) = C_R \cup [-R, -\varepsilon] \cup C_\varepsilon \cup [\varepsilon, R],$$

where

$$C_\rho = \{z = \rho e^{i\theta} \mid 0 \le \theta \le \pi\} \quad (\rho = \varepsilon,\ R),$$

$$[-R, -\varepsilon] = \{z = re^{i\pi} \mid \varepsilon \le r \le R\},$$

$$[\varepsilon, R] = \{z = r \mid \varepsilon \le r \le R\}.$$

We define an integral path by orienting $C(\varepsilon, R)$ using the counter-clockwise orientation. Calculate the value of the complex integral

$$\int_{C(\varepsilon,R)} f(z)\,dz.$$

(3) Show that $\displaystyle\int_{[-R,-\varepsilon]} f(z)\,dz = -e^{i\pi\alpha} \int_{[\varepsilon,R]} f(z)\,dz.$

(4) Calculate the value of the definite integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^2}\,dx.$$

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4 Consider a topological space X. For any subset S of X, we denote its closure by $\overline{S}$. We equip the Cartesian product $X \times X = \{(x, y) \mid x, y \in X\}$ with the product topology, i.e., $W \subset X \times X$ is an open set if for any $(x, y) \in W$ there exist open sets U, V of X such that $x \in U$, $y \in V$ and $U \times V \subset W$ hold. Here we consider the following 3 conditions about the topological space X:

(i) For any two distinct points $x, y \in X$, there exist open sets U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ hold.

(ii) For any $x \in X$, $\{x\} = \bigcap \left\{ \overline{U} \mid U \text{ is an open set containing } x \right\}$ holds.

(iii) The set $\Delta = \{(x, y) \in X \times X \mid x = y\}$ is a closed set of $X \times X$.

Answer the following questions.

(1) Show that Conditions **(i)** and **(ii)** are equivalent.

(2) Show that Conditions **(i)** and **(iii)** are equivalent.

(3) Consider continuous maps $f, g : X \to X$. Show that when X satisfies Condition **(i)**, the set $A = \{x \in X \mid f(x) = g(x)\}$ is a closed set of X.