Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2023 Admission

Part 1 of 2

February 4, 2023, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled **1**, **2**, **3**, and **4**, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, **3**, and **4** on pages **1**, **2**, **3**, and **4** of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 For real numbers a, b, define the matrix A and the vector $\boldsymbol{b}$ as follows.

$$A = \begin{pmatrix} a & -a & 0 & 0 \\ a & 0 & 2-a & -1 \\ -a & a & a-2 & 1 \\ 2a & -2a & -a+2 & -1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} -1 \\ 2 \\ b \\ -1 \end{pmatrix}$$

Answer the following questions about the system of linear equations

$$A\boldsymbol{x} = \boldsymbol{b}$$

in $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$.

(1) Find the rank of the matrix A.

(2) Find necessary and sufficient condition(s) on a, b so that this system of linear equations has a solution.

(3) When a, b satisfy the condition(s) from (2), find the general form of solutions of this system of linear equations.

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2 Consider the matrix A defined using a real number t as follows.

$$A = \begin{pmatrix} t+3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ -t-4 & t-2 & -1 & -3 \\ -t-1 & 0 & 0 & -1 \end{pmatrix}$$

Answer the following questions.

(1) Find the eigenvalues of A.

(2) Give the Jordan normal form of A. If necessary consider several cases depending on the value of t. You do not need to find an invertible matrix P such that $P^{-1}AP$ is in Jordan normal form.

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3 Answer the following questions about the functions $g(x, y) = x^2 + xy + y^2 - 1$ and $f(x, y) = xy$ on $\mathbb{R}^2$.

(1) Draw the graph of $g(x, y) = 0$.

(2) At a point $(x, y) = (a, b)$ other than the origin $(x, y) = (0, 0)$, find all $\lambda \in \mathbb{R}$ such that $\lambda \nabla g = \nabla f$ holds, where

$$\nabla g(x, y) = \begin{pmatrix} g_x(x, y) \\ g_y(x, y) \end{pmatrix}, \qquad \nabla f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}.$$

(3) Find the minimum and maximum values of f under the condition $g(x, y) = 0$, and find all the points that have these values.

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4 Answer the following questions.

(1) Let $u = u(x, y)$ be a real-valued function of class C^2 defined on $\mathbb{R}^2$. Let θ be a real number. For an arbitrary point (x, y) of $\mathbb{R}^2$, express the limit

$$\lim_{t \to 0} \frac{u(x + t\cos\theta, y + t\sin\theta) + u(x - t\cos\theta, y - t\sin\theta) - 2u(x, y)}{t^2}$$

using θ and the second partial derivatives of u.

(2) Define the function $f = f(x, y, z)$ on $\mathbb{R}^3$ as follows.

$$f(x, y, z) = (x^2 + y^2 + z^2)^{\frac{1}{2}}\left\{1 + (x^2 + y^2 + z^2)^{\frac{1}{2}} + z\right\}$$

For a parameter $\alpha > 0$, consider the improper integral

$$\int_{x^2+y^2+z^2>1} f(x, y, z)^{-\alpha}\, dxdydz.$$

Using a change to polar coordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta \quad (r \geq 0,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi),$$

rewrite this integral as an integral with respect to (r, θ, φ), and give necessary and sufficient condition(s) on $\alpha > 0$ so that it converges.