Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2023 Admission

Part 1 of 2

July 30, 2022, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and seat number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let a be a real number and consider the following 3×3 real matrix

$$A = \begin{pmatrix} a & 0 & a(1-a) \\ 0 & 2-a & a(1-a) \\ 0 & 0 & 2+a \end{pmatrix}.$$

(1) Find the eigenvalues of A.

(2) Show that A is diagonalizable.

(3) For a 3×3 real matrix M, let $V(M)$ be the real vector space consisting of all 3×3 real matrices that commute with M. Show that $\dim V(A) = \dim V(B)$, where B is a diagonal matrix obtained in (2).

(4) Find $\dim V(A)$.

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2 Let f be a linear map of $\mathbb{R}^4$ given by the 4×4 real matrix

$$A = \begin{pmatrix} 3 & 1 & 3 & -1 \\ 1 & 2 & 0 & -2 \\ 2 & -1 & 3 & 1 \\ 1 & -3 & 3 & 3 \end{pmatrix}.$$

Assuming that $\mathbb{R}^4$ is given the standard inner product, answer the following questions.

(1) For each of the image $\operatorname{Im} f$ and kernel $\operatorname{Ker} f$ of the linear map f, find its dimension and a basis.

(2) Let t be a real number. Let V be the subspace of $\mathbb{R}^4$ generated by the two vectors

$$\boldsymbol{v}_1 = \begin{pmatrix} 5 \\ 0 \\ 6 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} t+3 \\ t \\ -t+2 \\ -t \end{pmatrix}.$$

Find the dimension of W, where W is the orthogonal complement of V in $\mathbb{R}^4$.

(3) Let $f|_W : W \to \mathbb{R}^4$ denote the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$ restricted to W. Find the dimensions of $\operatorname{Ker}(f|_W)$ and $\operatorname{Im}(f|_W)$.

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3 (1) Show that, in a neighborhood of $(x, y, z) = (1, -1, 1+\sqrt{2})$, a class-1 function $z = f(x, y)$ of two variables satisfying

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{x} = 1$$

can be uniquely determined. In addition, compute the values of $\dfrac{\partial f}{\partial x}(1, -1)$ and $\dfrac{\partial f}{\partial y}(1, -1)$.

(2) Compute the value of the improper integral

$$\iint_D x^3 e^{-y} \frac{\sin y}{y^2}\, dxdy, \quad D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq x^2\}.$$

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4 Let $u = u(x,t)$ be the function defined by

$$u(x,t) = t^{-\frac{1}{2}} e^{-\frac{x^2}{t}} \qquad (x \in \mathbb{R},\, t > 0)$$

on $\mathbb{R} \times (0, \infty)$. Let $\partial_t^k u$ denote the k-th order partial derivative of the function u with respect to the variable t.

(1) Find the first-order partial derivative of u with respect to t.

(2) Show that, for any positive integer k, there exists a polynomial p_k of degree k such that

$$\partial_t^k u(x,t) = t^{-k} p_k\left(\frac{x^2}{t}\right) u(x,t)$$

for every $x \in \mathbb{R}$ and every $t > 0$.

(3) Fix a positive integer k. Show that there exists a constant C such that the inequality

$$\int_{-\infty}^{\infty} |\partial_t^k u(x,t)|\, dx \le C\, t^{-k}$$

holds for every $t > 0$.