Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2021 Admission

Part 2 of 2

February 7, 2021, 10:00 〜12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 3 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 3 problems labeled **1**, **2**, and **3** respectively. Please **answer all 3 problems.**
4. The answering sheet consists of 3 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems **1**, **2**, and **3** on pages **1**, **2**, and **3** of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 3 pages in the answering sheet.**
7. The back side of the 3 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 3 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Consider an $n \times n$ real matrix A such that $A^2 = {}^tA$, where ${}^tA$ denotes the transpose of A. For any elements $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ of $\mathbb{C}^n$, let $\langle x, y \rangle$ denote the standard Hermitian inner product $\sum_{j=1}^{n} x_j \overline{y_j}$. Answer the following questions.

(1) Let v be an eigenvector of A corresponding to an eigenvalue α. Show that $\alpha^2 = \overline{\alpha}$ by considering $\langle v, Av \rangle$.

(2) Show that eigenvectors of A corresponding to distinct eigenvalues are orthogonal.

Below we consider $n = 3$ and assume that A satisfies the following conditions:

" $A^2 = {}^tA$ holds, A is invertible and is not the identity matrix. "

(3) Show that at least one of the eigenvalues of A is real, and calculate all the eigenvalues of A. Also show that A is diagonalizable by a unitary matrix.

(4) Give an example of a matrix A satisfying the conditions.

2 Define $I = [0, 1]$. A function φ defined on I is said to be of bounded variation on I if there exists $M > 0$ such that

$$\sum_{j=0}^{n-1} |\varphi(t_{j+1}) - \varphi(t_j)| \leq M$$

holds for any partition

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

of I. Answer the following questions.

(1) Consider a function f that is differentiable on an open interval including I and such that its derivative f' is bounded on I. Show that f is of bounded variation on I.

(2) Determine whether the following functions are of bounded variation on I or not. Explain the reasons.

(i) $g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$ (ii) $h(x) = \begin{cases} x \cos \dfrac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$

3 Answer the following questions.

(1) Consider a real-valued function $f\colon U \to \mathbb{R}$ over an open set U ($\neq \emptyset$) of $\mathbb{R}^n$ that is of class C^1 and such that its partial derivatives $\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \ldots, \dfrac{\partial f}{\partial x_n}$ are identically zero on U. Show that f is a constant function if U satisfies the following condition:

"any two points of U can be connected by a curve of class C^1 inside U."

(2) A real-valued function $f\colon X \to \mathbb{R}$ over a topological space X ($\neq \emptyset$) is a locally constant function if for any $p \in X$ there exists an open set V containing p such that f is a constant function on V. Show that if X is connected, then a locally constant function on X is a constant function on X.