# Entrance Examination for Master's Program
# Graduate School of Mathematics
# Nagoya University
# 2020 Admission

## Part 2 of 2

February 6, 2020, 13:00 ∼16:00

**Note:**

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

**Notation:**

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

**1** Consider the function

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & (x > 0), \\ 0 & (x \leqq 0) \end{cases}$$

defined on $\mathbb{R}$.

(1) Show that $f(x)$ is differentiable at $x = 0$.

(2) Show that, for $x > 0$, the $n$-th derivative $f^{(n)}(x)$ of $f(x)$ can be expressed as

$$f^{(n)}(x) = p_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}}$$

using a polynomial $p_n(t)$. Furthermore, determine the degree of $p_n(t)$.

(3) Show that, for an arbitrary positive integer $n$, the function $f(x)$ is $n$ times differentiable at $x = 0$.

**2** For a linear transformation $S : V \to V$ on a finite dimensional real vector space $V$, let $T : V \to V$ be another linear transformation given by

$$T(\mathbf{v}) = S(\mathbf{v}) + \mathbf{v} \qquad (\mathbf{v} \in V).$$

Also, let $I$ be the identity transformation on $V$ and let $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$ be the kernel and the image of $T$, respectively.

(1) Show that if $S \circ S = I$, then $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$.

(2) Suppose that $S \circ S = I$, $S \neq \pm I$. Furthermore, let $(v_1, \cdots, v_m)$ and $(v_{m+1}, \cdots, v_n)$ be bases of $\mathrm{Ker}(T)$ and $\mathrm{Im}(T)$, respectively. Find the representation matrix of $S$ with respect to the basis $(v_1, \cdots, v_m, v_{m+1}, \cdots, v_n)$ of $V$.

(3) Suppose that $V = \mathrm{Ker}(T) \oplus \mathrm{Im}(T)$. Determine whether or not the statement $S \circ S = I$ holds. If it does, then prove it. Otherwise, show that it is false by finding a counterexample.

**3** Consider the complex function

$$f(z) = \frac{e^{3iz} - 3e^{iz} + 2}{z^3},$$

where $i$ is the imaginary unit. Also, for $R > \varepsilon > 0$, let $I_{(\varepsilon,R)} \cup \Gamma_R \cup J_{(\varepsilon,R)} \cup C_\varepsilon$ be the closed curve oriented counterclockwise, where

$$\Gamma_R = \{Re^{i\theta} \in \mathbb{C}\,;\, 0 \leqq \theta \leqq \pi\},$$

$$C_\varepsilon = \{\varepsilon e^{i\theta} \in \mathbb{C}\,;\, 0 \leqq \theta \leqq \pi\},$$

$$I_{(\varepsilon,R)} = \{x \in \mathbb{R}\,;\, \varepsilon \leqq x \leqq R\},$$

$$J_{(\varepsilon,R)} = \{x \in \mathbb{R}\,;\, -R \leqq x \leqq -\varepsilon\}.$$

(1) Compute $\displaystyle\lim_{R\to+\infty} \int_{\Gamma_R} f(z)\,dz$.

(2) Compute $\displaystyle\lim_{\varepsilon\to 0} \int_{C_\varepsilon} f(z)\,dz$.

(3) Compute the improper integral

$$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx.$$

**4** Consider a metric space $(X, d)$. For an arbitrary nonempty subset $A \subseteqq X$, let $d_A : X \to [0, +\infty)$ be the function defined by

$$d_A(x) = \inf\{d(x, a)\,;\, a \in A\} \qquad (x \in X).$$

(1) For arbitrary $x, y \in X$, show that

$$|d_A(x) - d_A(y)| \leqq d(x, y).$$

(2) Suppose that $A$ is a closed set. Determine $\{x \in X\,;\, d_A(x) = 0\}$ and justify your answer.

(3) Suppose that $A$ and $B$ are both nonempty closed subsets of $X$ and $A \cap B = \emptyset$. Show that there exists a function $f : X \to \mathbb{R}$ so that

$$d_A(x) = (d_A(x) + d_B(x))f(x)$$

for all $x \in X$.

(4) Suppose that $(X, d)$ is connected. Determine the range $\{f(x)\,;\, x \in X\}$ of the function $f$ considered in (3) and justify your answer.