Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2020 Admission

Part 1 of 2

February 6, 2020, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Consider the two systems of linear equations below, where a, b, c, d are real constants and x, y, z, w are unknowns:

$$\begin{cases} 2x - y - z + 4w = -3 \\ 3x - 2y + z + 3w = -1 \\ x \phantom{{}- 2y + z} + aw = b \end{cases} \qquad \begin{cases} x \phantom{{}+y} + z + w = 3 \\ x + y \phantom{{}+cz} + 3w = 4 \\ x \phantom{{}+y} + cz \phantom{{}+3w} = d. \end{cases}$$

(1) Determine the values of a, b, c, d so that the two systems have two or more common solutions.

(2) Obtain the common solutions in (1).

(3) Determine the necessary and sufficient condition(s) for a, b, c, d so that there is no common solution to the two systems.

2 Answer the following questions.

(1) For a constant $c > 0$, compute the improper integral $\displaystyle\int_{-\infty}^{\infty} x^n e^{-cx^2}\,dx$ $(n = 0, 1, 2)$.

(2) Diagonalize the matrix $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ using an orthogonal matrix.

(3) Let A be the matrix in (2). Compute the matrix $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ determined by the improper double integral

$$b_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_j \exp\langle A\mathbf{x}, \mathbf{x}\rangle\, dx_1 dx_2, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$(i, j = 1, 2)$. Here, $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2$ for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$.

3 Answer the following questions. Each of (1), (2), (3) is an independent question.

(1) Find a polynomial $p(x)$ satisfying the inequality

$$|e^x - p(x)| \leqq 10^{-1} \quad (-1 \leqq x \leqq 1).$$

You may use the fact that $e < 3$.

(2) Compute the improper double integral $\displaystyle\int_0^\infty \left(\int_y^\infty x^2 e^{-x^2} \, dx \right) dy$.

(3) Find all critical points of the function $f(x, y) = (2x^2 + y^2)e^{-x^2-y^2}$ defined on the 2-dimensional Euclidean space $\mathbb{R}^2$ and determine whether or not f takes an extremum at each of these points.

4 The set V consisting of all sequences of real numbers is a real vector space with respect to:

$$\begin{array}{lll} \text{addition:} & \{x_n\}_{n=1}^{\infty} + \{y_n\}_{n=1}^{\infty} = \{x_n + y_n\}_{n=1}^{\infty}, & \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in V, \\ \text{scalar multiplication:} & \alpha\{x_n\}_{n=1}^{\infty} = \{\alpha x_n\}_{n=1}^{\infty}, & \alpha \in \mathbb{R},\ \{x_n\}_{n=1}^{\infty} \in V. \end{array}$$

Let W be the subset of V consisting of all sequences of real numbers satisfying the recurrence relation

$$(*) \qquad x_{n+3} - 6x_{n+2} + 11x_{n+1} - 6x_n = 0 \quad (n = 1, 2, \ldots).$$

(1) Show that W is a subspace of V and determine its dimension.

(2) Let $S : V \to V$ be the linear transformation given by

$$S(\{x_n\}_{n=1}^{\infty}) = \{x_{n+1}\}_{n=1}^{\infty}.$$

Show that $S(W) \subseteq W$ and determine all eigenvalues of the linear transformation $T : W \to W$, the restriction of S to W.

(3) Give a basis of W so that the representation matrix of the linear transformation T in (2) is a diagonal matrix.

(4) Assuming $x_1 = 1$, $x_2 = 0$, $x_3 = 2$, find the general term x_n of the sequence $\{x_n\}_{n=1}^{\infty}$ of real numbers satisfying the recurrence relation $(*)$.