Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2020 Admission

Part 1 of 2

July 27, 2019, 9:00 $\sim$12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let $t \in \mathbb{R}$ and consider the following vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_t = \begin{pmatrix} 1 \\ 0 \\ t \\ -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{d}_t = \begin{pmatrix} 1 \\ 1 \\ 1 \\ t \end{pmatrix}$$

in the vector space $\mathbb{R}^4$. Let V_t and W_t be the subspaces spanned by $\mathbf{a}, \mathbf{b}_t$ and $\mathbf{c}, \mathbf{d}_t$, respectively.

(1) Compute the dimension of $V_t + W_t$.

(2) Determine the value(s) of t for which the dimension of $V_t \cap W_t$ does not equal 0. In addition, give a basis for $V_t \cap W_t$.

(3) Determine the value(s) of t for which the vector $\mathbf{x} = \begin{pmatrix} 0 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ can be written as a sum of a vector in V_t and a vector in W_t. In addition, write $\mathbf{x}$ as a sum of a vector in V_t and a vector in W_t.

2 Consider the 3×3 matrix

$$A = \begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix},$$

where $a \in \mathbb{R}$ and $b > 0$.

(1) Find the eigenvalues of A.

(2) Diagonalize A and compute A^n.

(3) Compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} A^n = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{1}{n!} A^n$. Here, the limit of a sequence of 3×3 matrices is the 3×3 matrix each of whose entries equals the limit of the corresponding entries of the matrices in the sequence.

3 Answer the following questions. Each of (1), (2), (3) is an independent problem.

(1) Let p, q be positive real numbers. Determine whether the improper integral $\displaystyle\int_0^\infty \frac{dx}{(x^p + 2019)^q}$ is convergent or divergent.

(2) Suppose that a real-valued function $f(x, y)$ of class C^1 defined on the 2-dimensional Euclidean space $\mathbb{R}^2$ satisfies

$$y\,\frac{\partial f}{\partial x}(x, y) - x\,\frac{\partial f}{\partial y}(x, y) = 0, \quad (x, y) \in \mathbb{R}^2.$$

Show that $f(x, y)$ can be written as

$$f(x, y) = g(r), \qquad r = \sqrt{x^2 + y^2}$$

for some real-valued function $g(t)$ of one variable.

(3) For the rational function $\left(\dfrac{1+x}{1-x}\right)^2$, give the Taylor series expansion at 0.

4 Let $p, q \in \mathbb{R}$ with $p^2 + q^2 = 1$. In the 2-dimensional Euclidean space $\mathbb{R}^2 = \{(x, y)\,;\, x, y \in \mathbb{R}\}$, consider the closed region

$$D = \left\{ (x, y) \in \mathbb{R}^2 \,;\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leqq 1 \right\} \quad (0 < a < b)$$

enclosed by an ellipse and the line $\ell_{p,q}$ determined by the equation

$$px + qy = 0.$$

(1) Compute $\displaystyle\iint_D x^2\,dxdy$ and $\displaystyle\iint_D y^2\,dxdy$.

(2) Show that $\displaystyle\iint_D xy\,dxdy = 0$.

(3) Let $r_{p,q}(x, y)$ be the distance between a point $(x, y) \in D$ and the line $\ell_{p,q}$ (that is, the length of the line segment which joins (x, y) to $\ell_{p,q}$ and is perpendicular to $\ell_{p,q}$). Find the values of p, q for which

$$I(p, q) = \iint_D r_{p,q}(x, y)^2\,dxdy$$

is minimized.