Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2019 Admission

Part 2 of 2

February 6, 2019, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Define the convergence of a vector in the complex vector space $\mathbb{C}^k$ by the convergence of each component of the vector.

(1) Let A be a 3×3 Jordan block $\begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$ $(\alpha \in \mathbb{C})$. Find A^n $(n \geq 2)$.

(2) Consider A given in (1). Find the necessary and sufficient condition on α so that the sequence $\{A^n\mathbf{x}\}_{n=0}^{\infty}$ of vectors in $\mathbb{C}^3$ converges for an arbitrary $\mathbf{x} \in \mathbb{C}^3$.

(3) Let B be a $k \times k$ complex matrix. Find the necessary and sufficient condition on B so that the sequence $\{B^n\mathbf{x}\}_{n=0}^{\infty}$ of vectors in $\mathbb{C}^k$ converges for an arbitrary $\mathbf{x} \in \mathbb{C}^k$.

2 Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^n$ $(n \geq 1)$. An $n \times n$ real symmetric matrix $A = (a_{ij})$ is said to be positive definite if $\langle A\mathbf{x}, \mathbf{x} \rangle \geq 0$ for an arbitrary $\mathbf{x} \in \mathbb{R}^n$ and $\langle A\mathbf{x}, \mathbf{x} \rangle = 0$ implies that $\mathbf{x} = \mathbf{0}$. Also, an $n \times n$ real symmetric matrix $A = (a_{ij})$ is positive semidefinite if $\langle A\mathbf{x}, \mathbf{x} \rangle \geq 0$ for an arbitrary $\mathbf{x} \in \mathbb{R}^n$. Suppose that each of $A = (a_{ij})$ and $B = (b_{ij})$ is an $n \times n$ positive definite real symmetric matrix and $A - B = (a_{ij} - b_{ij})$ is positive semidefinite.

(1) For a positive definite real symmetric matrix C, let $V(C) = \{\mathbf{x} \in \mathbb{R}^n \mid \langle C\mathbf{x}, \mathbf{x} \rangle < 1\}$. Show that

$$V(A) \subset V(B).$$

(2) Express the volume

$$\int_{V(A)} 1 \, dx_1 dx_2 \cdots dx_n$$

of $V(A)$ in terms of the volume ω_n of the unit ball $\{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle < 1\}$ in $\mathbb{R}^n$ and $\det(A)$. There is no need for calculating the value of ω_n.

(3) Show that $\det(A) \geq \det(B)$.

3 (1) Show that $z = n \in \mathbb{Z}$ is a pole of the meromorphic function

$$\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$$

on $\mathbb{C}$. Show also that, for each n, the order and residue at the pole $z = n$ are both 1.

(2) For a positive integer n, let C_n be the circumference of the square in the complex plane with its four corners at the points $\pm(n + \frac{1}{2}) \pm (n + \frac{1}{2})i$. Show that there exists a constant M so that, for each positive integer n, $|\cot(\pi z)| \leq M$ on C_n.

(3) For the function $f(z)$ given by

$$f(z) = \frac{1}{1 + z^2},$$

show that

$$\lim_{n \to +\infty} \int_{C_n} f(z) \pi \cot(\pi z) dz = 0$$

holds, where C_n is that considered in (2) oriented counterclockwise.

(4) Compute the limit

$$\lim_{n \to \infty} \sum_{k=-n}^{n} \frac{1}{1 + k^2}.$$

4

(1) Using sequences of points, define that a subset A of a metric space (X, d) is a closed set.

(2) Let (X, d) be a compact metric space and consider the monotone decreasing sequence

$$X \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset A_{n+1} \supset \cdots$$

of nonempty closed subsets. Show that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

(3) If (X, d) is a metric space that is not compact, does the same conclusion as in (2) hold? If it does, prove it. Otherwise, give a counterexample and show that it is indeed a counterexample.