# Entrance Examination for Master's Program
# Graduate School of Mathematics
# Nagoya University
# 2019 Admission

## Part 1 of 2

February 6, 2019, 9:00 ～12:00

**Note:**

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

**Notation:**

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

**1** Let $a, b, c$ be real-valued parameters. Let $f$ be a linear transformation of $\mathbb{R}^3$ defined by the $3 \times 3$ real matrix

$$A = \begin{pmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{pmatrix}.$$

(1) Find the necessary and sufficient condition on $a, b, c$ so that the rank of $f$ equals 1.

(2) Find the necessary and sufficient condition on $a, b, c$ so that

$$\det(A) = 0 \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \operatorname{Im}(f).$$

**2** Let $A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}$.

(1) Find the eigenvalues of the matrix $A$ and their (algebraic) multiplicity.

(2) Find the eigenspace of the matrix $A$ associated with each eigenvalue.

(3) For an integer $n \geq 2$, find the matrix $A^n$.

**3** Each of (1), (2), (3) is an independent problem.

(1) Let $\alpha$ be a positive constant. Show that the equation $\lim_{x \to +\infty} x^{\alpha} e^{-x} = 0$ holds.

(2) Let $n$ be a positive integer and $z \in \mathbb{C}$. Expand the function

$$F(z) = \frac{1}{1 + z + z^2 + \cdots + z^n}$$

into a power series around $z = 0$ and find the radius of convergence.

(3) Let $f$ be a function of class $C^2$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ satisfying

$$f_{rr} = f_{\theta\theta}$$

in polar coordinates. Show that the function

$$E(r) = \int_0^{2\pi} (f_r^2 + f_\theta^2)\, d\theta$$

on $\mathbb{R}_+ = \{r \in \mathbb{R} | r > 0\}$ is a constant function.

**4** (1) For a real number $p > 1$, show that the limit

$$\lim_{b \to +\infty} \int_1^b \frac{\cos t}{t^p} dt$$

converges. There is no need for calculating the value of the limit.

(2) Show that the limit

$$\lim_{b \to +\infty} \int_0^b \sin(x^2) dx$$

converges. There is no need for calculating the value of the limit.