Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2019 Admission

Part 2 of 2

July 28, 2018, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 For $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, let $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \bar{y}_i$ be the standard Hermitian inner product in $\mathbb{C}^n$.

(1) Show that an $n \times n$ complex matrix A with the condition (I) is an Hermitian matrix:

(I) For an arbitrary $\mathbf{x} \in \mathbb{C}^n$, $\langle A\mathbf{x}, \mathbf{x} \rangle$ is a real number.

(2) Show that an $n \times n$ complex matrix B with the condition (II) is invertible:

(II) For an arbitrary $\mathbf{x} \in \mathbb{C}^n$, $\langle B\mathbf{x}, \mathbf{x} \rangle$ is a real number and, in addition, $\langle B\mathbf{x}, \mathbf{x} \rangle = 0$ implies $\mathbf{x} = \mathbf{0}$.

(3) Show that $n \times n$ complex matrices C, D with the condition (III) satisfy $\operatorname{rank} C \geq \operatorname{rank} D$:

(III) For an arbitrary $\mathbf{x} \in \mathbb{C}^n$, both $\langle C\mathbf{x}, \mathbf{x} \rangle$ and $\langle D\mathbf{x}, \mathbf{x} \rangle$ are real numbers and, in addition, $\langle C\mathbf{x}, \mathbf{x} \rangle \geq \langle D\mathbf{x}, \mathbf{x} \rangle \geq 0$.

2 Determine whether or not each of the following statements holds. If it does, then prove it. Otherwise, show that it is false by finding a counterexample.

(1) If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

(2) If $\{a_n\}_{n=1}^{\infty}$ is a sequence satisfing $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

(3) Supposes that $\{a_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence with $a_n > 0$ ($\forall n = 1, 2, \dots$). If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

(4) If a positive term series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$. (A positive term series is a series whose terms are nonnegative.)

3 Answer the following questions.

(1) For $x \in [0, \pi]$, $y \geq 0$, $(x, y) \neq (0, 0)$, write each of the real part and imaginary part of $\log(\sin(x + iy))$ using only real numbers. (For the complex logarithm, take the principal value.)

(2) Let x, y be real numbers. Compute the limit

$$\lim_{y \to \infty} \left(\int_0^{\pi} \{\log(\sin(x + iy)) - y\} dx \right).$$

(Again, take the principal value for the complex logarithm.)

(3) Compute the integral

$$\int_0^{\pi} \log(\sin x)\, dx.$$

4 Let $f(x)$ be a continuous function defined on the half-open interval $(0, 1]$. Define a subset $I \subset \mathbb{R}$ as follows: $a \in I$ if and only if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in the interval $(0, 1]$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) = a$.

(1) Obtain I when $f(x) = \sin \frac{1}{x}$. No explanation is needed.

(2) Show that, in general, if I is nonempty, then it is a connected closed set.