Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2019 Admission

Part 1 of 2

July 28, 2018, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Answer the following questions.

(1) Show that the three vectors $\begin{pmatrix}1\\2\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-2\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\2\end{pmatrix}$ in $\mathbb{R}^4$ are linearly independent.

(2) Show that the three vectors $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\-1\\0\end{pmatrix}$ in $\mathbb{R}^4$ are linearly independent.

(3) Let V_1 and V_2 be the subspaces in $\mathbb{R}^4$ spanned by the three vectors in (1) and (2), respectively. Find the dimension of the subspace $V_1 + V_2$ in $\mathbb{R}^4$.

(4) For the subspaces V_1 and V_2 in (3), determine the dimension of $V_1 \cap V_2$ and find its basis.

2 Consider the matrix

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and answer the following questions.

(1) Find a unitary matrix U such that $U^{-1}TU$ is a diagonal matrix.

(2) Show that the 3×3 complex matrices that commute with T form a 3-dimensional complex vector space.

(3) Find all 3×3 complex matrices that commute with each of T and

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Clearly justify your answer.

3 Answer the following questions. Each of (1), (2), (3) is an independent problem.

(1) Express

$$F(z) = \frac{1}{1+z+z^2}$$

as a power series in z centered at the origin and find its radius of convergence.

(2) Let $n \geq 2$ and A be a real symmetric $(n \times n)$ matrix. Define a function $f(\mathbf{x})$ on $\mathbb{R}^n$ by $f(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product. Express the differential mapping

$$\mathbb{R}^n \ni \mathbf{y} \mapsto (Df)_{\mathbf{x}}(\mathbf{y}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{y}) - f(\mathbf{x})}{h} \in \mathbb{R}$$

in terms of A. In addition, show that a point $\mathbf{x}$ that maximizes f on $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}$ is an eigenvector of A.

(3) Compute the double integral

$$\int_0^\infty \int_0^\infty e^{-(s^2 - st + t^2)} ds dt.$$

Hint : transform $s^2 - st + t^2$ into complete square.

4 Answer the following questions.

(1) For $0 < \theta \leq 1$, compute the limit

$$\lim_{n\to\infty}\left[\max_{x\geq n}\left\{\left(1+\frac{x}{n}\right)^n e^{-\theta x}\right\}\right].$$

(2) Show that

$$\lim_{n\to\infty}\int_n^\infty \left(1+\frac{x}{n}\right)^n e^{-x}\,dx = 0.$$