Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2018 Admission

Part 2 of 2

July 29, 2017, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be a linear space and $f, g\colon V \to V$ be linear maps. We assume that $f \circ g = g \circ f$.

(1) Show that $\mathrm{Ker}(f \circ g) \supset \mathrm{Ker}\, f + \mathrm{Ker}\, g$.

(2) Show that $g(\mathrm{Ker}\, f) \subset \mathrm{Ker}\, f$.

It follows from (2) that the restriction of g to $\mathrm{Ker}\, f$ defines a linear map $\tilde{g}\colon \mathrm{Ker}\, f \to \mathrm{Ker}\, f$.

(3) Suppose that $\tilde{g}$ is injective. Then, prove that $\mathrm{Ker}\, f \cap \mathrm{Ker}\, g = \{0\}$.

(4) Suppose that $\tilde{g}$ is surjective. Then, prove that $\mathrm{Ker}(f \circ g) = \mathrm{Ker}\, f + \mathrm{Ker}\, g$.

2 Answer the following questions. You can use without proof that (a) a continuous function is integrable on a finite closed interval, and that (b) when a series of continuous functions converges locally uniformly, the limit is a continuous function.

(1) Let f be a real valued function of class C^1 on $\mathbb{R}$. We define a series $\{a_n\}_{n=1}^{\infty}$ by

$$a_n = \int_0^{2\pi} f(x) \sin nx \, dx.$$

Then, show that there is a constant C such that

$$|a_n| \leq \frac{C}{n}, \quad n = 1, 2, \ldots.$$

(2) Let $\{f_n(x)\}_{n=1}^{\infty}$ be a series of real valued functions of class C^1 on $\mathbb{R}$ which satisfies $f_n(0) = 0$. Suppose that $f_n'(x)$ converges locally uniformly to $g(x)$ on $\mathbb{R}$. Then, prove that $f_n(x)$ converges locally uniformly to $\displaystyle\int_0^x g(y)\, dy$ on $\mathbb{R}$.

(3) Let $C > 0$ be a constant, and assume that a series of real numbers $\{a_k\}_{k=1}^{\infty}$ satisfies the following

$$|a_k| \leq \frac{C}{k^3}, \quad k = 1, 2, \ldots$$

Show that the series of functions

$$f_n(x) = \sum_{k=1}^{n} a_k \sin kx, \quad n = 1, 2, \ldots.$$

converges uniformly to a function of class C^1 on $\mathbb{R}$.

3 Let $a, b > 0$. Consider the following function

$$f(z) = \frac{ze^{ibz}}{z^2 + a^2}, \quad z \in \mathbb{C}.$$

Answer the following questions.

(1) Find all the poles of $f(z)$ and the associated residues.

(2) Let $R > 0$. We define Γ_R to be the following half circle in the complex plane oriented counterclockwise.

$$\left\{ z \in \mathbb{C} \,\middle|\, |z| = R, \operatorname{Im} z \geq 0 \right\}.$$

Show that

$$\lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz = 0.$$

(3) Compute the value of the limit $\displaystyle\lim_{R \to \infty} \int_{-R}^{R} \frac{x \sin bx}{x^2 + a^2}\, dx$.

4 Let X, Y be topological spaces. We consider the family of subsets of the direct product $X \times Y$ defined by

$$\mathcal{B} = \{A \times B \mid A \text{ is an open set in } X,\ B \text{ is an open set in } Y\}.$$

The topology of $X \times Y$ is called the product topology, if it has $\mathcal{B}$ as its open basis. In the following, we equip $X \times Y$ with the product topology. Further, we define a map $f : X \times Y \to X$ by $f(x, y) = x$ $(x \in X,\ y \in Y)$.
State if each of the following statements is true or false. If true, prove it. If false, give a counter-example and prove that it is indeed a counter example.

(1) f is continuous.

(2) If G is an open set in $X \times Y$, then, $f(G)$ is an open set in X.

(3) If F is a closed set in $X \times Y$, then, $f(F)$ is a closed set in X.