Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2018 Admission

Part 1 of 2

February 6, 2018, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let t be a real number. Consider the linear mapping $f_t : \mathbb{R}^4 \to \mathbb{R}^4$ given by $f_t(u) = A_t u$, where A_t is the matrix shown below:

$$A_t = \begin{pmatrix} 1 & -2 & 1 & 1 \\ 4t & 3-2t & -3-2t & 4t \\ 3-8t & -3+4t & 4t & 3-8t \\ 3-2t & -3+t & t & 3-2t \end{pmatrix}.$$

(1) Find k, where k is the dimension of $\mathrm{Ker} f_t$. Also, find a basis $u_1, ..., u_k$ of $\mathrm{Ker} f_t$ such that none of u_j $(j = 1, ..., k)$ depends on t.

(2) Find t such that the dimension of $\mathrm{Ker} f_t + \mathrm{Im} f_t$ equals 3.

2 For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, ..., y_n) \in \mathbb{R}^n$, let $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. Also, assume that $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ and $w_1, w_2, \ldots, w_n \in \mathbb{R}^n$ satisfy the condition $\langle v_i, w_j \rangle = \delta_{ij} \quad (1 \leq i, j \leq n)$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

(1) Show that $w_1, w_2, \ldots, w_n$ form a basis of $\mathbb{R}^n$.

(2) Suppose that A is an $n \times n$ matrix satisfying

$$Ax = \sum_{i=1}^n w_i \langle v_i, x \rangle$$

for all $x \in \mathbb{R}^n$. Find A.

(3) Let $n = 3$. Suppose that B is a 3×3 matrix satisfying

$$Bx = w_2 \langle v_1, x \rangle + w_3 \langle v_2, x \rangle + w_1 \langle v_3, x \rangle$$

for all $x \in \mathbb{R}^3$. Find the eigenvalues of B. Also, for each eigenvalue, express corresponding eigenvectors in terms of w_1, w_2, w_3.

3 Assume that each of (1) and (2) is an independent problem.

(1) Determine the necessary and sufficient condition on $p > 0$ so that the following improper integral converges:

$$\int_0^1 x^{-p} \sin \frac{1}{x}\, dx.$$

(2) Let $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, x, y, z \geq 0, x + y + z \leq 1\}$. Find the value of the following triple integral.

$$\int_D (x + y + z + 1)^{-3}\, dxdydz$$

4 For $x > 0$, let $f_n(x) = \sum_{j=1}^{n} \frac{x}{1+j^2x^2}$ $(n = 1, 2, ...)$.

(1) For an arbitrary $x > 0$, show that $f(x) = \lim_{n\to\infty} f_n(x)$ exists and f satisfies the following inequality:

$$\frac{\pi}{2} - \text{Arctan}\, x \leq f(x) \leq \frac{\pi}{2} - \text{Arctan}\, x + \frac{x}{1+x^2}.$$

(2) Determine whether or not the convergence $f_n(x) \to f(x)$ $(n \to \infty)$ is uniform for $x > 0$ and give a proof for the result.