Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2017 Admission

Part 1 of 2

July 30, 2016, 9:00 ∼12:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled 1, 2, 3, and 4, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems 1, 2, 3, and 4 on pages 1, 2, 3, and 4 of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let a, b, c, d be real numbers. The subsets V and W of $\mathbb{R}^4$ are given by

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{lllll} x_1 & +2x_2 & +x_3 & +4x_4 & = c \\ x_1 & +3x_2 & & +4x_4 & = c \\ x_1 & +x_2 & +ax_3 & +4x_4 & = c \end{array} \right\},$$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{lllll} x_1 & +3x_2 & -x_3 & +4x_4 & = d \\ 2x_1 & +(b+5)x_2 & -2x_3 & +(2b+6)x_4 & = d \end{array} \right\}.$$

(1) Find the necessary and sufficient condition on a, b, c, d so that $V \cap W$ is a linear subspace of $\mathbb{R}^4$.

For (2) and (3), assume that a, b, c, d satisfy the condition described in (1).

(2) Find the necessary and sufficient condition on a, b, c, d so that $\dim(V \cap W) = 1$ and give a basis of $V \cap W$.

(3) Find the necessary and sufficient condition on a, b, c, d so that $V \oplus W = \mathbb{R}^4$.

2 Let V be the complex linear space spanned by all 3×3 matrices with complex entries. Also, let $\mathbb{C}[t]$ be the set of all polynomials in t with complex coefficients. For a matrix

$$A = \begin{pmatrix} 0 & -\alpha\beta & 0 \\ 1 & \alpha+\beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, consider the set

$$W = \{f(A) \mid f(t) \in \mathbb{C}[t]\}.$$

(1) Show that W is a complex linear subspace of V.

(2) Determine the dimension of W.

(3) Find the necessary and sufficient condition on α, β, γ so that the dimension of W is minimum. Furthermore, find a Jordan canonical form of A when the dimension of W is indeed minimum. (There is no need to find a matrix that transforms A into a Jordan canonical form.)

3 Assume that each of (1), (2), (3) is an independent problem.

(1) Let $f(x, y)$ be a differentiable function on $\mathbb{R}^2$. Perform a change of variables by

$$x = \frac{\sin\theta}{\cos\varphi}, \quad y = \frac{\sin\varphi}{\cos\theta} \quad (\theta, \varphi > 0,\ \theta + \varphi < \pi/2)$$

and consider the function g in θ, φ defined by $g(\theta, \varphi) = f\left(\frac{\sin\theta}{\cos\varphi}, \frac{\sin\varphi}{\cos\theta}\right)$. Write each of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ using $\theta, \varphi, \frac{\partial g}{\partial \theta}, \frac{\partial g}{\partial \varphi}$.

(2) Let $p \in \mathbb{R}$. Determine the necessary and sufficient condition on p so that the following improper integral converges.

$$\int_0^\infty \frac{x^2 + 1 - \cos x}{(x^2+1)x^p} dx.$$

(3) Let f, g be functions of class C^1 defined on the interval $[0, 1]$ with $g(0)g(1) \neq 0$. Write the following double integral I using $f(0), f(1), g(0), g(1)$.

$$I = \iint_{[0,1]\times[0,1]} f(x)f'(x)g'(y)\exp(f(x)g(y))dxdy.$$

4 Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers satisfying

$$a_n > 0, \quad n = 0, 1, \ldots$$

and

$$a_n \geq \frac{1}{2}(a_{n+1} + a_{n-1}), \quad n = 1, 2, \ldots .$$

For each $n = 1, 2, \ldots$, let $b_n = a_n - a_{n-1}$.

(1) Show that $\{b_n\}_{n=1}^{\infty}$ is a monotone non-increasing sequence.

(2) Show that $\{b_n\}_{n=1}^{\infty}$ is bounded below. If necessary, use a proof by contradiction.

(3) Show that $\lim_{n\to\infty} \dfrac{a_n}{n}$ converges.