Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2017 Admission

Part 2 of 2

February 7, 2017, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V and W be finite dimensional linear spaces over $\mathbb{R}$, and $f : V \to W$ be a linear map.

(1) Show that

$$\dim V = \dim \mathrm{Ker} f + \dim \mathrm{Im} f$$

by proving the following (i) and (ii).

(i) Let $v_1, \ldots, v_k$ be a basis of $\mathrm{Ker} f$, and $w_1, \ldots, w_m$ be a basis of $\mathrm{Im} f$. We choose $v_{k+1}, \ldots, v_{k+m} \in V$ such that

$$f(v_{k+i}) = w_i, \quad i = 1, \ldots, m.$$

Then prove that $v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m}$ are linearly independent.

(ii) Prove that $v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m}$ is a basis of V.

(2) Let U be a finite dimensional linear space over $\mathbb{R}$, and $g : W \to U$ be a linear map. Prove that the composition map $g \circ f$ satisfies

$$\dim \mathrm{Im}(g \circ f) \geq \dim \mathrm{Im} f + \dim \mathrm{Im} g - \dim W.$$

2 Let $v_1, v_2 \in \mathbb{R}^2$ be linearly independent vectors, and $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that

$$f(x + v_1) = f(x + v_2) = f(x)$$

for any $x \in \mathbb{R}^2$.

(1) For any $x \in \mathbb{R}^2$, show that there exist integers $m_1(x)$ and $m_2(x)$ such that

$$x - m_1(x)v_1 - m_2(x)v_2 \in \{t_1 v_1 + t_2 v_2 \in \mathbb{R}^2 \mid t_1, t_2 \in [0, 1]\}.$$

(2) Show that f attains a maximum and a minimum over $\mathbb{R}^2$.

3 (1) Let R be a positive real number. Assume that a power series with complex coefficients $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on

$$D_R = \{z \in \mathbb{C} \mid |z| < R\}.$$

Write a_n by using a complex integral involving $f(z)$ along the circle in the complex plane centered at the origin with radius r. Here r satisfies $0 < r < R$.

(2) Let $g(z)$ be a holomorphic function on $\mathbb{C}$. For any real number ρ, we define

$$M(\rho) = \max_{z \in \mathbb{C},\, |z| = \rho} |g(z)|.$$

Assume that there exist a positive constant M and a positive integer k such that

$$M(\rho) \leq M\rho^k$$

for any $\rho > 0$. Then find all possible $g(z)$.

4 Let (X, d) be a metric space with metric d. When a subset A of X satisfies the following condition, it is said to be sequentially compact.

> For any sequence $\{x_n\}_{n=1}^{\infty}$ in A, there exist a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and a point a of A such that
>
> $$\lim_{j\to\infty} d(x_{n_j}, a) = 0.$$

(1) Assume that A is a sequentially compact subset of X. Is A a closed set? If so, give a proof. Otherwise, give a counterexample and show that it is indeed a counterexample.

In the following, we assume that (X, d_X) and (Y, d_Y) are metric spaces, and that $f : X \to Y$ is continuous.

(2) Prove that if A is a sequentially compact subset of X, then $f(A)$ is a sequentially compact subset of Y.

(3) Let A_m $(m = 1, 2, \dots)$ be sequentially compact subsets of X such that

$$A_1 \supset A_2 \supset A_3 \supset \cdots .$$

Then prove that

$$\bigcap_{m=1}^{\infty} f(A_m) = f\Big(\bigcap_{m=1}^{\infty} A_m\Big).$$