Entrance Examination for Master's Program
Graduate School of Mathematics
Nagoya University
2017 Admission

Part 2 of 2

July 30, 2016, 13:00 ∼16:00

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let A and B be $n \times n$ diagonalizable matrices with complex entries.

(1) Show that, if there is a matrix that diagonalizes both A and B, then $AB = BA$.

For (2) and (3), assume that $AB = BA$. Also, for an eigenvalue α of A, let $W(A, \alpha)$ be the corresponding eigenspace.

(2) For an arbitrary vector $v \in W(A, \alpha)$, show that $Bv \in W(A, \alpha)$.

(3) (i) Show that there is a basis for $W(A, \alpha)$ consisting of eigenvectors of B.

(ii) Using (i), show that there is a matrix that diagonalizes both A and B.

2 Suppose that f is a nonnegative continuous function defined on the interval $[0, \infty)$ such that

$$\lim_{x \to \infty} f(x) = \alpha$$

for some $\alpha > 0$.

(1) Show that f is bounded on $[0, \infty)$.

(2) Show that the improper integral $\displaystyle\int_0^\infty f(x)\,dx$ diverges.

Next, suppose that g is a nonnegative continuous function defined on the interval $[0, \infty)$ such that

$$\int_0^\infty g(x)\,dx < \infty.$$

(3) Show that there exists a sequence $\{a_n\}_{n=1}^\infty \subset [0, \infty)$ satisfying

$$\lim_{n \to \infty} a_n = \infty \quad \text{and} \quad \lim_{n \to \infty} g(a_n) = 0.$$

If necessary, consider the above improper integral in terms of

$$I_n = \int_n^{n+1} g(x)\,dx \quad (n = 0, 1, 2, \dots).$$

(4) Does $\displaystyle\lim_{x \to \infty} g(x) = 0$ hold? If so, give a proof. Otherwise, give a counterexample and show that it is indeed a counterexample.

3 Let $a > 0$ be a constant and consider the complex function

$$f(z) = \frac{1}{(z^2 + a^2)\sin(\pi z)}.$$

(1) Find all singularities of $f(z)$ in $\mathbb{C}$.

(2) Let N be a natural number with $N > a$. Let C_N be the circle in the complex plane centered at the origin with radius $N + \frac{1}{2}$. The circle C_N is given by the counter-clockwise orientation. Find the value of the integral

$$\int_{C_N} f(z)\,dz.$$

(3) Find the value of the series $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + a^2}$. If necessary, the fact

$$|\sin(\pi z)| > \frac{1}{\sqrt{2}}, \quad z \in C_N,\ N = 1, 2, \ldots$$

may be used.

4

(1) Define that a subset K of a topological space X is compact in terms of open cover.

(2) Suppose that a topological space X is compact and a subset A of X is a closed set. Is A compact? If so, give a proof. Otherwise, give a counterexample and show that it is indeed a counterexample.

(3) Define that a topological space Y satisfies the Hausdorff separation axiom in terms of open set.

(4) Suppose that a topological space Y satisfies the Hausdorff separation axiom and a subset B of Y is compact. Is B a closed set? If so, give a proof. Otherwise, give a counterexample and show that it is indeed a counterexample.