Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2016 Admission

Part 2 of 2

Thursday, February 4, 2016, 13:00 p.m.∼16:00 p.m.

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let A be an $n \times n$ matrix with complex entries. Suppose that A has 0 as an eigenvalue with multiplicity $m \geq 1$.

(1) Let k be the number of the Jordan blocks with eigenvalue 0 in a Jordan canonical form of A. Show that $k \leq m$.

(2) Show that the following inequality holds.

$$n - m \leq \operatorname{rank} A \leq n - 1.$$

(3) Show that, if $\operatorname{rank} A = \operatorname{rank} A^2$, then $\operatorname{rank} A = n - m$.

2 Consider the function

$$f(x) = \frac{1}{x^2}$$

defined on the open interval $I = (0, 1)$ in $\mathbb{R}$.

(1) Give an ε-δ proof of the fact that the function $f(x)$ is continuous at an arbitrary point $a \in I$.

(2) Determine whether or not $f(x)$ is uniformly continuous on I.

Here, a function $f(x)$ defined on an interval $I \subset \mathbb{R}$ is uniformly continuous if the following condition is satisfied.

For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in I,\ |x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

3 Let $a \in \mathbb{R}$ and consider the complex function

$$f(z) = \frac{e^{az}}{z^2+1}.$$

For a real number $R > 100$, let

$$C_R = \{z = 1 + Re^{i\theta} \in \mathbb{C} \mid \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\},$$

$$L_R = \{z = 1 + iy \in \mathbb{C} \mid -R \leq y \leq R\}.$$

Furthermore, give an orientation to each of C_R and L_R so that the closed curve $C_R \cup L_R$ is traversed counterclockwise.

(1) Obtain the value of the complex integral $\displaystyle\int_{C_R \cup L_R} f(z)dz$.

(2) Assuming $a > 0$, obtain the values of $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)dz$ and $\displaystyle\lim_{R\to\infty} \int_{L_R} f(z)dz$.

(3) Assuming $a \leq 0$, obtain the value of $\displaystyle\lim_{R\to\infty} \int_{L_R} f(z)dz$.

4 Let (X, d) be a metric space. For $a, b \in X$, let $d(a, b)$ denote the distance between a and b. For the following questions, you may assume the completeness of $\mathbb{R}$.

(1) For two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ of points in X, show the following inequality.

$$|d(a_m, b_m) - d(a_n, b_n)| \leq d(a_m, a_n) + d(b_m, b_n).$$

(2) Suppose that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are both Cauchy sequences of points in X. Show that, if $(c_n)_{n=1}^{\infty}$ is the sequence in $\mathbb{R}$ given by $c_n = d(a_n, b_n)$, then $(c_n)_{n=1}^{\infty}$ converges.

(3) Let Y be the set of all Cauchy sequences of points in X. If we define a relation $\sim$ on Y by

$$(a_n)_{n=1}^{\infty} \sim (b_n)_{n=1}^{\infty} \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(a_n, b_n) = 0,$$

then $\sim$ is an equivalence relation on Y.

For the quotient set $\overline{Y} = Y/\sim$ of Y under the equivalence relation $\sim$, define a map $\overline{d} : \overline{Y} \times \overline{Y} \to \mathbb{R}$ by

$$\overline{d}(\overline{(a_n)_{n=1}^{\infty}}, \overline{(b_n)_{n=1}^{\infty}}) = \lim_{n\to\infty} d(a_n, b_n).$$

Show that $\overline{d}$ is well-defined.