Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2016 Admission

Part 1 of 2

Thursday, February 4, 2016, 9:00 a.m.～12:00 noon

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$, respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ on pages $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the 4 pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4 page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let W_1 and W_2 be the subspaces of the vector space $\mathbb{R}^4$ over $\mathbb{R}$ given by

$$W_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; \begin{array}{r} x_1 - 3x_2 - x_3 + 2x_4 = 0, \\ 2x_1 - 3x_2 - 2x_3 + x_4 = 0 \end{array} \right\},$$

$$W_2 = \left\{ t \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; t, s \in \mathbb{R} \right\}.$$

(1) Obtain a basis for W_1.

(2) Obtain a basis for the sum $W_1 + W_2$.

(3) Find a tuple a, b, c, d, e, f, g, h of real numbers for which the solution space of the linear system

$$\begin{cases} ax_1 + bx_2 + cx_3 + dx_4 = 0, \\ ex_1 + fx_2 + gx_3 + hx_4 = 0 \end{cases}$$

is exactly W_2.

2 Let V be a finite-dimensional vector space over $\mathbb{R}$ with an inner product $(\ ,\ )$. A linear map $f : V \to V$ is said to be symmetric if f satisfies

$$(f(v), w) = (v, f(w)), \quad v, w \in V.$$

(1) Suppose that a linear map $f : V \to V$ is symmetric and has the inverse f^{-1}. Show that f^{-1} is also symmetric. (There is no need for showing that f^{-1} is a linear map.)

(2) Let $e_1, \ldots, e_n$ be an orthonormal basis of V and consider a linear map $f : V \to V$. Show that, if f is symmetric, then the representation matrix A of f with respect to the basis $e_1, \ldots, e_n$ is a symmetric matrix.

(3) For an arbitrary linear map $f : V \to V$, show that there is a unique linear map $g : V \to V$ such that

$$(f(v), w) = (v, g(w)), \quad v, w \in V.$$

3 For the real function

$$f(x) = \frac{1}{1 - x - x^2},$$

let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ be the Taylor expansion of $f(x)$ at $x = 0$.

(1) Find a_0, a_1.

(2) For each $n = 0, 1, 2, \ldots,$ find a_n.

(3) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

4 Let S be the surface in $\mathbb{R}^3$ given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + (y - z)^2 + z^2 = 1\}.$$

(1) Show that S is a bounded closed set.

(2) Let $F(x, y, z, \lambda)$ be the real function of four variables given by

$$F(x, y, z, \lambda) = 2yz + \lambda\{x^2 + (y - z)^2 + z^2 - 1\}.$$

Also, let F_x, F_y, F_z be the partial derivatives of F with respect to x, y, z, respectively. By taking λ as a constant, the equation

$$F_x(x, y, z, \lambda) = F_y(x, y, z, \lambda) = F_z(x, y, z, \lambda) = 0$$

can be seen as a system of linear equations involving x, y, z. Find all real numbers λ for which the above equation has a solution other than the trivial solution $(x, y, z) = (0, 0, 0)$.

(3) Find all points (x, y, z) on S at which the function $f(x, y, z) = 2yz$ takes its maximum. In addition, find the maximum value of f.