Entrance Examination for the Ph. D. Program
Graduate School of Mathematics
Nagoya University
2015 Admission

Part 2 of 2

Saturday, July 26, 2014, 13:00 p.m.∼16:00 p.m.

Note:

1. Please do not turn pages until told to do so.
2. The problem sheet consists of the cover page and 4 single-sided pages. After the exam has begun, please first confirm that the number of pages and their printing and order are correct. Please report any problem immediately.
3. There are a total of 4 problems labeled [1], [2], [3], and [4], respectively. Please **answer all 4 problems.**
4. The answering sheet consists of 4 single-sided pages. Please **confirm the number of pages**, and please **do not remove the staple**.
5. Please write the answers to problems [1], [2], [3], and [4] on pages [1], [2], [3], and [4] of the answering sheet, respectively.
6. **Please write name and application number in the space provided on each of the** 4 **pages in the answering sheet.**
7. The back side of the 4 pages in the answering sheet may also be used. If used, please check the box at the lower right-hand corner on the front side.
8. If the answering sheet staple is torn, or if additional paper is needed for calculations, please notify the exam proctor.
9. After the exam has ended, please hand in the 4-page answering sheet. The problem sheet and any additional sheets used for calculations may be taken home.

Notation:

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

1 Let V be an n-dimensional real vector space with an inner product $(\ ,\ )$. Consider an arbitrary basis $\{a_1, \ldots, a_n\}$ of V.

(1) Let $\{e_1, \ldots, e_n\}$ be an arbitrary orthonormal basis of V. For the two bases $\{e_1, \ldots, e_n\}$ and $\{a_1, \ldots, a_n\}$, let $P = (p_{ij})$ be the transformation matrix, that is,

$$a_j = \sum_{i=1}^{n} p_{ij} e_i.$$

If $C = (c_{ij})$ is the $n \times n$ matrix given by

$$c_{ij} = (a_i, a_j),$$

then express C using P. Furthermore, show that $\det C > 0$.

(2) Show that there exists a unique n-element subset $\{b_1, \ldots, b_n\}$ of V such that

$$(a_i, b_j) = \delta_{ij} \quad \text{(where } \delta_{ij} \text{ is the Kronecker's delta).}$$

(3) Show that $\{b_1, \ldots, b_n\}$ in (2) is a basis of V. Also, show that $\det D = (\det C)^{-1}$, where $D = (d_{ij})$ is the $n \times n$ matrix given by

$$d_{ij} = (b_i, b_j).$$

2 Suppose that $\varphi(x, y)$ is a continuous function on $\mathbb{R}^2$ satisfying

$$\iint_{\mathbb{R}^2} \varphi(x, y)\, dxdy = 1, \quad \varphi(x, y) \geq 0.$$

Let $\varphi_t(x, y) = t^{-2}\varphi(t^{-1}x, t^{-1}y)$ for each $t > 0$. Also, let $f(x, y)$ be a continuous function on $\mathbb{R}^2$ that is identically 0 outside a bounded set $K \subset \mathbb{R}^2$.

(1) Fine the value of the double integral

$$\iint_{\mathbb{R}^2} \varphi_t(x, y)\, dxdy.$$

(2) For each $\delta > 0$, find the right-hand limit

$$\lim_{t \to +0} \iint_{x^2+y^2 \geq \delta^2} \varphi_t(x, y)\, dxdy.$$

(3) Explain why $f(x, y)$ is bounded on $\mathbb{R}^2$.

(4) Prove that the equation

$$\lim_{t \to +0} \iint_{\mathbb{R}^2} \varphi_t(x, y) f(x, y)\, dxdy = f(0, 0)$$

holds.

3 For $R > 0$, let Γ_R be the boundary of D_R traversed counterclockwise, where D_R is the semidisk

$$D_R = \{z \in \mathbb{C} \,|\, |z| \leq R,\, \operatorname{Im} z \geq 0\}$$

in the complex plane. Here, $\operatorname{Im} z$ denotes the imaginary part of z.

(1) Let α, β, $\gamma \in \mathbb{C}$, none of which belongs to Γ_R. Find the conditions on α, β, γ for which the complex integral

$$\int_{\Gamma_R} \frac{dz}{(z-\alpha)(z-\beta)(z-\gamma)}$$

equals 0.

(2) Let α, β, $\gamma \in \mathbb{C}$, none of which is on the real axis. Find the conditions on α, β, γ for which the improper integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\alpha)(x-\beta)(x-\gamma)}$$

equals 0.

4 Let A be a subset of a topological space X. A point x in A is called an interior point of A if there exists an open set U of X such that $x \in U$ and $U \subset A$. Let A° and $A^\times$ be the subsets of X defined by

$$A^\circ : \text{ the set of interior points of } A,$$

$$A^\times : \text{ the union of all open sets contained in } A.$$

(1) Show that $A^\circ = A^\times$.

(2) For subsets $A_1, \ldots, A_n$ of X, prove that the following holds.

$$\left(\bigcap_{i=1}^{n} A_i \right)^\circ = \bigcap_{i=1}^{n} A_i^\circ.$$

(3) Give an example of a topological space X and an infinite family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subsets of X for which

$$\left(\bigcap_{\lambda \in \Lambda} A_\lambda \right)^\circ \neq \bigcap_{\lambda \in \Lambda} A_\lambda^\circ.$$