1 Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map given by

$$f(u) = Au \qquad (u \in \mathbb{R}^4)$$

for

$$A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 4 & 6 & -5 & 1 \\ 3 & 1 & -2 & -1 \\ 0 & -2 & 1 & -1. \end{pmatrix}$$

(1) Determine the dimension and find a basis of Ker f.

(2) Determine the dimension and find a basis of Im f.

(3) For a given real number t, let W_t be the subvectorspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{pmatrix} 1 \\ t \\ 3 \\ -2 \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ -2 \\ -5 \\ t \end{pmatrix}.$$

Find the values of t for which the subspace (Im f) $+ W_t$ is a 3-dimensional subvectorspace.

2 Let ${}^tA$ be the transpose of the matrix A. Given an integer $n \geq 2$, we say that a symmetric real $n \times n$-matrix A is postive definite if ${}^tuAu > 0$ for any $u \in \mathbb{R}^n \setminus \{0\}$.

(1) Show that for any real regular matrix B, the matrix $A = {}^tBB$ is a positive definite, symmetric real matrix.

(2) Show that for a positive definite, symmetric real matrix A, every eigenvalue is a postive real number.

(3) Show that for every positive definite, symmetric real matrix A, there is a real regular matrix B such that $A = {}^tBB$.

3

(1) Among all rectangular parallelepipeds whose sum of the length of the edges is 12, find one with maximal surface area.

(2) Show that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Use this to calculate, for given $a \in \mathbb{R}$ and $b > 0$, the value of the improper integral

$$\iint_D e^{-ax-by}\, dxdy, \qquad D = \{(x,y) \in \mathbb{R}^2 |\ y - x^2 > 0\}.$$

(3) Find the area of the cylinder surface $x^2 + y^2 = 4$ inside of the solid body

$$\{(x,y,z) \in \mathbb{R}^3 |\ |x| \le 2,\ |y| \le 2,\ 0 \le z \le 4 - x^2\}.$$

4

(1) Let f be the real valued function given by $f(x) = \log(1+x)$ on the interval $(-1, \infty)$. Find the Taylor expansion and its radius of convergence of f at $x = 0$ (Give a reason).

(2) For a positive integer n, let $a_n = \left(1 + \frac{1}{n}\right)^n$. Use the expansion of (1) to find the real number γ such that

$$\lim_{n\to\infty} n^\gamma (\log a_n - 1)$$

has a finite value different from 0. For such γ, calculate the limit.

(3) For $\{a_n\}$ and γ as in (2), find the limit

$$\lim_{n\to\infty} n^\gamma (a_n - e).$$