1 Let V be the real vector space of 2×2 square matrices with real entries, and consider the linear map $f : V \to V$ given by

$$f(X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} X \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad (X \in V).$$

Let

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(1) Show that $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis of V, and give the representing matrix of f with respect to this basis.

(2) Determine the dimension of the image $\mathrm{Im}\, f$.

(3) Give a basis of the kernel $\mathrm{Ker}\, f$.

2 For a given real 4×4 square matrix X, we define $X_{11}, X_{12}, X_{21}, X_{22}$ to be the 2×2 square matrices in a block decomposition of X as follows:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Let A, B, Z be real 4×4 matrices satisfying

$$A_{12} = O, \qquad A_{21} = O, \qquad A_{22} = O, \qquad B_{21} = O, \qquad \det(AZ + B) \neq 0,$$

where we write O for the 2×2 zero matrix.

(1) Show that $A_{11}Z_{11} + B_{11}$ and B_{22} are regular matrices.

(2) Let $S = (AZ + B)^{-1}$. Express S_{11}, S_{22} in terms of the blocks of A, B and Z.

(3) For real 4×4 square matrices C and D, let $T = (CZ + D)S$, where S is as in (2). Show that if $C_{12} = O$, then

$$T_{11} = (C_{11}Z_{11} + D_{11})(A_{11}Z_{11} + B_{11})^{-1}.$$

3 Consider the map $\Phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by

$$\Phi(r, t) = (r \cosh t, r \sinh t) \qquad (r > 0,\ t \in \mathbb{R}),$$

where

$$\cosh t = \frac{e^t + e^{-t}}{2}, \qquad \sinh t = \frac{e^t - e^{-t}}{2} \qquad (t \in \mathbb{R}).$$

(1) Show that for any fixed $r_0 > 0$, the point $(x, y) = \Phi(r_0, t)$ lies on the curve

$$C_{r_0} = \{(x, y) \in \mathbb{R}^2 |\ x^2 - y^2 = r_0^2,\ x \geq r_0\}.$$

(2) Show that $\Phi : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ is injective, and that the image of Φ is

$$D = \{(x, y) \in \mathbb{R}^2 |\ x + y > 0,\ x - y > 0\}.$$

(3) Fix $r_0 > 0$ and $t_0 > 0$, and let P_0 be the point $(x_0, y_0) = \Phi(r_0, t_0)$. Draw the region bounded by the line from the origin $(0, 0)$ to P_0, the x-axis, and the curve C_{r_0} given in (1), and calculate its area.

4 (1) Suppose that the sequence of real numbers $\{A_n\}$ either diverges $\lim_{n\to\infty} A_n = \infty$, or converges to a number larger than 1, $\lim_{n\to\infty} A_n > 1$. Then, show that the series $\sum_{n=1}^{\infty} \frac{1}{n^{A_n}}$ converges.

(2) For $\alpha \in \mathbb{R}$, give a function $f_\alpha(t)$ on $(0, \infty)$ such that the equation

$$(\log n)^{(\log n)^\alpha} = n^{f_\alpha(\log n)}$$

holds for any integer $n \geq 2$.

(3) Given $\alpha \in \mathbb{R}$, determine whether or not the series

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{(\log n)^\alpha}}$$

converges.